# Construction of Sparse Suffix Trees and LCE Indexes in Optimal Time and Space

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- Problems: small-space LCE indexes and sparse suffix trees

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- Known reduction to a locally consistent parsing
- Deterministic locally consistent parsing in small space

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Construct LCE index and SST in $O(b)$ space and $O(n)$ time?

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Given a $\tau$-partitioning set of size $O(b)$ with $\tau = \frac{n}{b}$, one can construct in $O(n)$ time an LCE index with $O(\frac{n}{b})$-time queries and an SST on any $b$ suffixes using $O(b)$ space on top of the input

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Main result

For $\tau = \frac{n}{b}$, a $\tau$-partitioning set of size $O(b)$ can be constructed in $O(n \log_b n)$ time using $O(b)$ space on top of the string $s[1..n]$

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Related: synchronizing sets, minimizers, locally consistent parsing...

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The devil is in details

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- Do $O(\log^* n)$ reductions until $v_h$ is $O(1)$ or $\infty$ for all $h$

# Deterministic coin tossing

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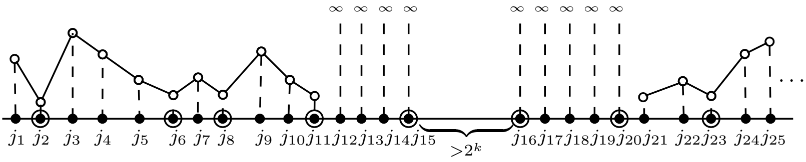
Put into $S_{k+1}$ all $j_h$ such that $j_h - j_{h-1} > 2^k$ or $j_{h+1} - j_h > 2^k$ or $\infty > v_{h-1} > v_h < v_{h+1}$ (local minima $v_h$)



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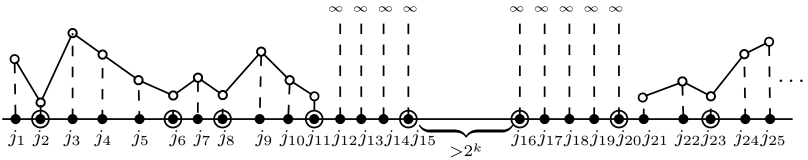


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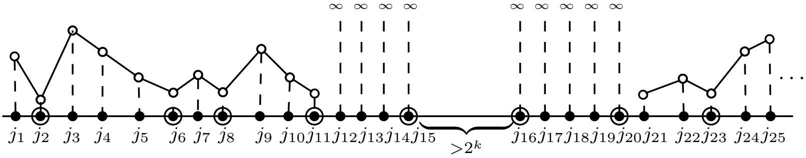


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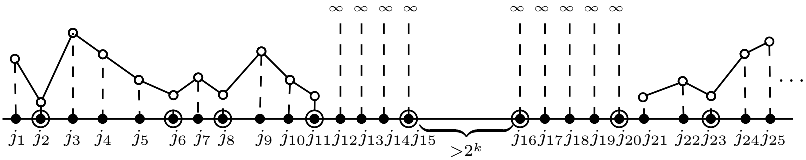


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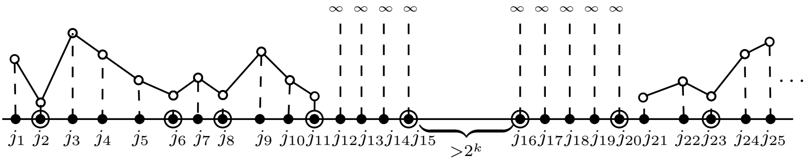


(Exact conditions are more complicated...)
How to store the sets $S_k$? Do we have to?

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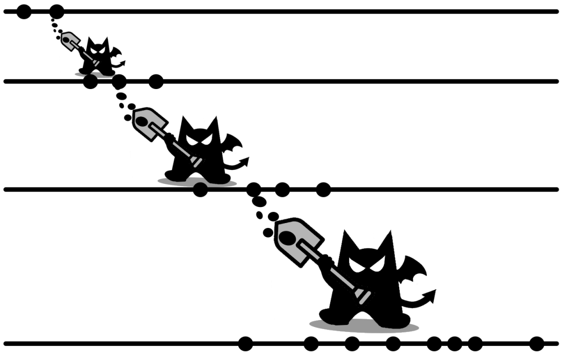
The decision to put $j \in S_k$ into $S_{k+1}$ is "local". We process $S_k$ left-to-right and feed the result to the same procedure processing $S_{k+1}$ left-to-right. The "cascade" of procedures feeding each other:



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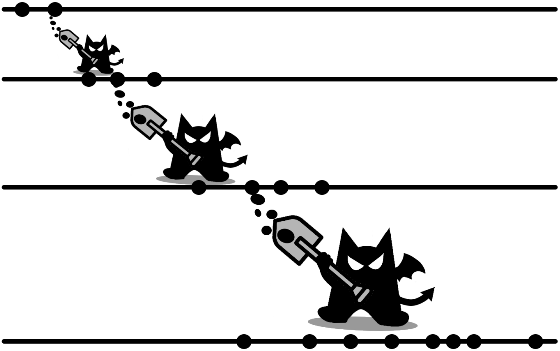

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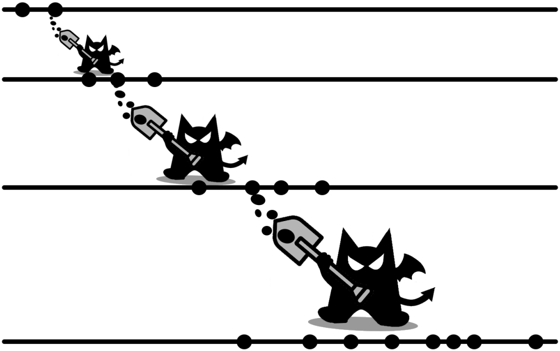

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The last level receives a $\tau$-partitioning set of size $O(\frac{n}{\tau} \log^* n)$

# To recompression

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The resulting set $S$ of size $O(\frac{n}{\tau}\log^* n)$ cannot be stored. Instead we make a string $R$ of length $|S|$ over a small alphabet which can be stored in $O(\frac{n}{\tau})$ machine words, such that any two letters of $R$ corresponding to positions of $S$ at a distance $\le \frac{\tau}{2}$ are distinct.

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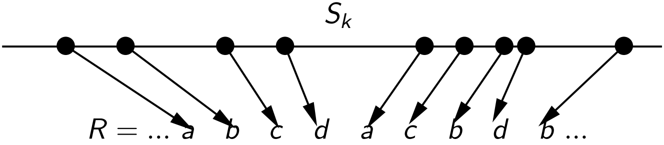

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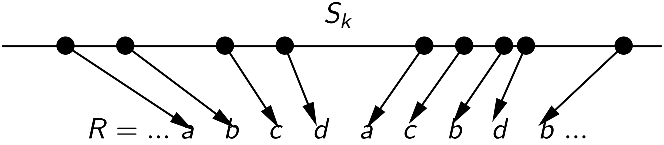


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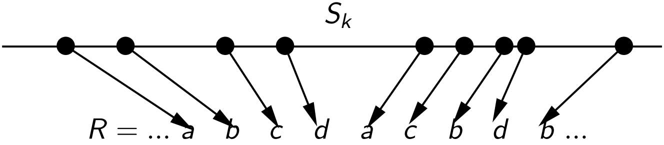


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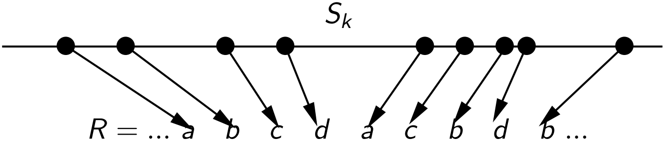


How the letters of $R$ are constructed?
Simple: interwined magic cascade of Vishkin–Cole magic!
We store separately the approximate info about distances between positions of $S$ sufficient to determine if $|j - j'| < \frac{\tau}{2}$ for $j, j' \in S$

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| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| a | b | c | d | a | c | b | d | b | a | c | d | b | a |

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| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | | 1 | 0 | | 0 | 1 | 1 | 0 | | 1 | 0 | 0 |
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| a | b | c | d | a | c | b | d | b | a | c | d | b | a |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| a | b | c | d | a | c | b | d | b | a | c | d | b | a |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | | 1 | 0 | | 0 | 1 | 1 | 0 | | 1 | 0 | 0 |
| a | b | c | d | a | c | b | d | b | a | c | d | b | a |

Generate the set $S$ using Vishkin–Cole again, retaining only those positions that correspond to remaining letters of $R$

# Thank you for your attention!