The *rational* construction of a (Wheeler) DFA

Giovanni Manzini, Alberto Policriti, Nicola Prezza, and Brian Riccardi

Order is important ...

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... and some orders are more important than others (e.g. the order of $\mathbb{Q}$)

Automata and (Co-lexicographic) Order

$\mathcal{A}$ *is input-consistent*: $(\forall u, v \in Q)(\delta(u, a_1) = \delta(v, a_2) \rightarrow a_1 = a_2)$.

$$\delta(q) = q' \text{ stands for } \delta(q, \lambda(q')) = q'.$$

Definition

A Wheeler DFA (WDFA) $\mathcal{A} = (Q, s, \delta, F, \prec)$ is such that $(Q, \prec)$ is a total order with s as minimum, and letting $v_1 = \delta(u_1)$, and $v_2 = \delta(u_2)$:

i $v_1 \prec v_2 \;\Rightarrow\; \lambda(v_1) \leqslant \lambda(v_2)$;

ii $(\lambda(v_1) = \lambda(v_2) \wedge v_1 \prec v_2) \Rightarrow u_1 \prec u_2$.

(Gagie-Manzini-Sirén. TCS 2017) (Alanko-D'Agostino-P.-Prezza. Inf. and Comp. 2021)

(Cotumaccio-D'Agostino-P.-Prezza. JACM. 2023)

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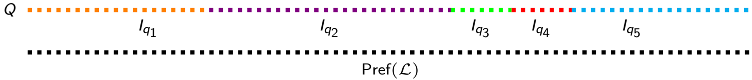
$\Sigma = \{a_1, a_2, \ldots, a_\sigma\}$ is ordered and W(i)-W(ii) extend the ordering to strings reaching states

co-lexicographically

align to the right and compare right-to-left (cbaabba $<$ aaacba)

WDFA: states are intervals of strings



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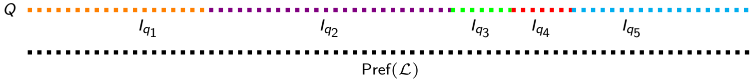
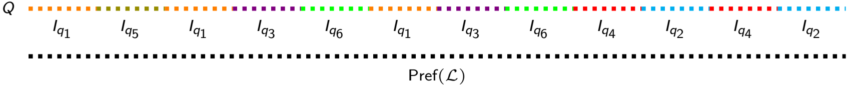


DFA



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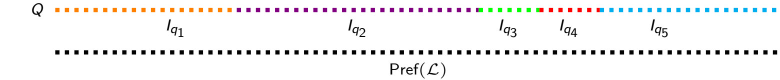

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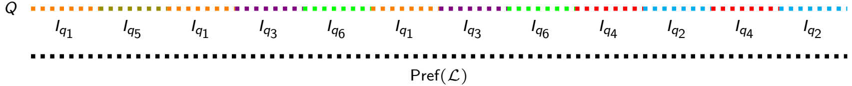
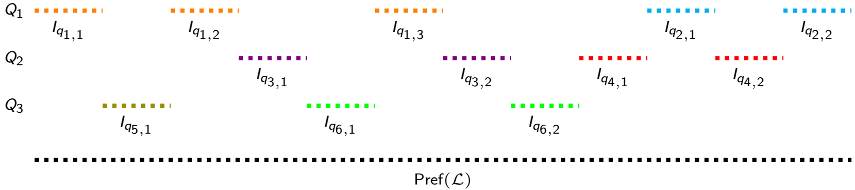
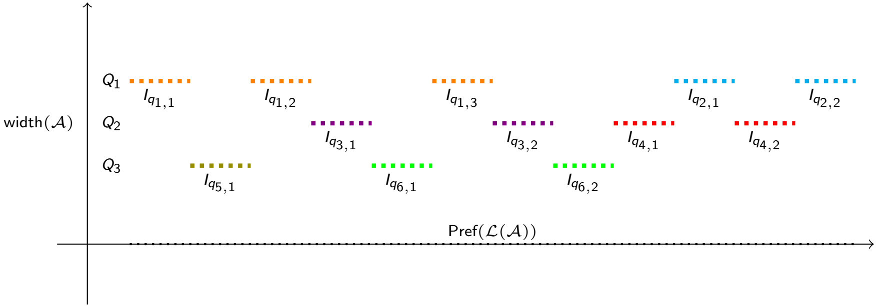




… coordinates



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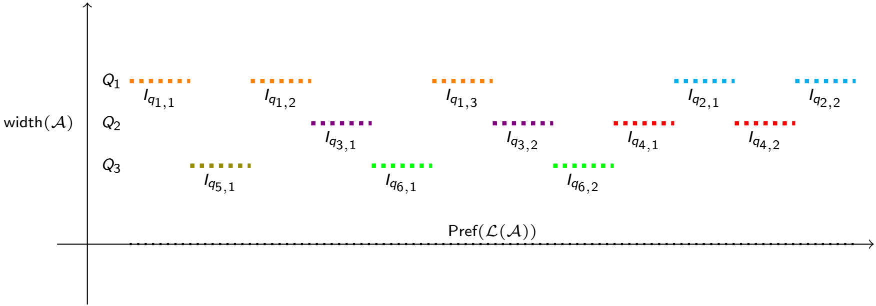


- (substring closure) membership in $\mathcal{L}(\mathcal{A})$ in $O(\text{width}(\mathcal{A})^2)$ per matched character;
- any NFA $\mathcal{N}$ is equivalent to a DFA $\mathcal{D}$ with at most $2^{\text{width}(\mathcal{N})}(|\mathcal{N}| - \text{width}(\mathcal{N}) + 1)$ states;
- any automaton of width p can be encoded in $O(\log p + \log|\Sigma|)$ bits per transition.

(Cotumaccio-D'Agostino-P.-Prezza. JACM. 2023)

Intermezzo: Path coherence

Definition (Path coherence)

Given a NFA $\mathcal{A}$ and an order $\prec$ over the set of its states, we say that $\mathcal{A}$ is *path coherent* iff strings send intervals (of states) into intervals:

$$\forall q \preceq q' \; \forall \alpha \; \exists p \preceq p' \; \big([q, q'] \overset{\alpha}{\rightsquigarrow} [p, p']\big).$$

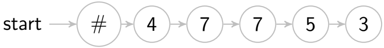
The rational embedding of strings

Definition (**The Rational Embedding of Σ^***)

The *Rational Embedding* of Σ^* is the map $\mathbb{q} : \Sigma^* \to \mathbb{Q}[0,1)$ such that, for any $\alpha = \alpha_1 \ldots \alpha_m \in \Sigma^*$:

$$\mathbb{q}(\alpha) = \sum_{i=1}^{m} \alpha_i \cdot (\sigma + 2)^{-(m-i+1)}.$$

Example



$\mathbb{q}(\alpha) = \mathbb{q}(47753) = 0.35774$

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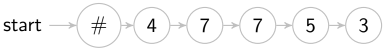

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property:
$\boldsymbol{\alpha} < \boldsymbol{\beta}$ (in co-lex order) $\Leftrightarrow \mathbb{q}(\boldsymbol{\alpha}) < \mathbb{q}(\boldsymbol{\beta})$ (as rational numbers).

The rational embedding of DFAs

Definition

$I_{\mathbb{Q}[0,1)}$ the collection of convex sets of rationals in $\mathbb{Q}[0,1)$:

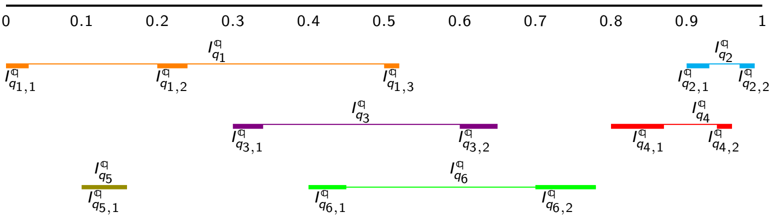
Definition (**The Rational Embedding of a DFA)**

The *Rational Embedding* of $\mathcal{A} = (Q, s, \delta, F)$ is the map $I^{\mathfrak{q}} : Q \to I_{\mathbb{Q}[0,1)}$ defined as follows: for any $q \in Q$,

$$I^{\mathfrak{q}}(q) = \bigcap \{J \in I_{\mathbb{Q}[0,1)} \mid (\forall \alpha \in I_q)(\mathfrak{q}(\alpha) \in J)\}.$$

$I^{\mathfrak{q}}(q)$ is the convex closure (hull) of I_q. (Notation: $I_q^{\mathfrak{q}} = I^{\mathfrak{q}}(q)$)

The rational embedding of DFAs



Determinism: $q \neq q' \Rightarrow I_q \cap I_{q'} = \emptyset$.
But it might be that $q \neq q' \wedge I^{\mathbb{Q}}_q \cap I^{\mathbb{Q}}_{q'} \neq \emptyset$.

From (Alanko-D'Agostino-P.-Prezza. Inf. and Comp. 2021)[Theorem 4.3]:

$\mathcal{A}$ is Wheeler iff $q \neq q' \Rightarrow I^{\mathbb{Q}}_q \cap I^{\mathbb{Q}}_{q'} = \emptyset$.

The rational embedding of DFAs

Example

There are *"solid"* collections of rational embeddings of strings accepted by a DFA: Σ^* (non-denumerable set of accumulation points)

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Missing largest and smallest digit $\Rightarrow$ (in general) there is a successor but no predecessor.

$$0.7348 < \cdots < 0.73488 < \cdots < 0.734888 < \cdots 0.7345 < 0.73451$$

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Question: are $I_q^{\mathbb{q}}$'s left and right limits (ℓ_q, r_q) always in $\mathbb{Q}$?

Ordering states: Entanglement

Definition

$Q' \subseteq Q$ is entangled if there exists a monotone sequence $(\alpha_i)_{i \in \mathbb{N}}$ in $Pref(\mathcal{L}(\mathcal{D}))$ such that:

$$\forall u' \in Q' \; \delta(s, \alpha_i) = u' \text{ for infinitely many } i's$$

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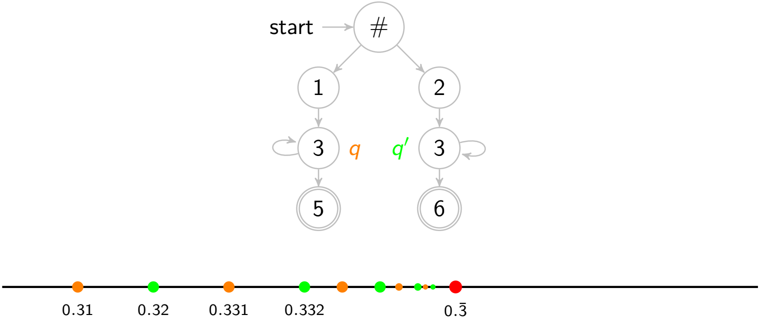
Lemma

If a value x is a left-accumulation point (resp. right-accumulation point) for both the sets $I_q^{\mathbb{Q}}$ and $I_{q'}^{\mathbb{Q}}$ then q and q' are entangled.

Examples



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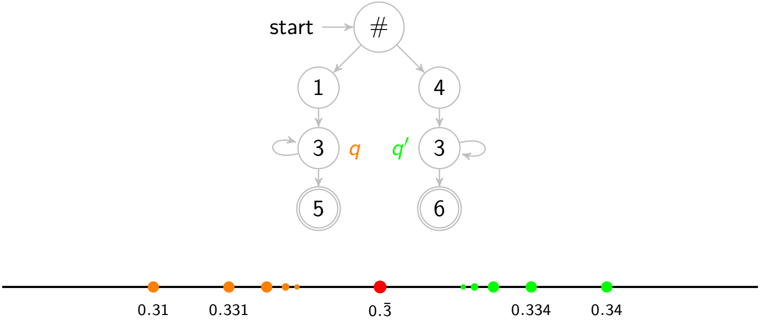

Finding Left and Right Limits

Lemma

Let $\mathcal{L} = \mathcal{L}(\mathcal{D})$, with $\mathcal{L}$ Wheeler and $\mathcal{D}$ either minimum or Wheeler. For any $q \in Q$ we have:

$$\ell_q = 0.a_{q,1} \cdots a_{q,h}\overline{a_{q,h+1} \cdots a_{q,h+j}},$$

with $h + j \leqslant |Q|$, and $j > 0$ if and only if $\ell_q \notin I_q^{\mathfrak{q}}$.

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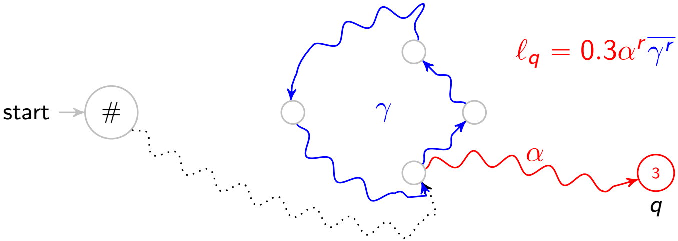
Proof's idea

walk backward from q and find α whose rational embedding $\mathbb{q}(\alpha)$ is the smallest among those reaching q.



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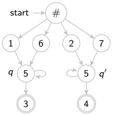
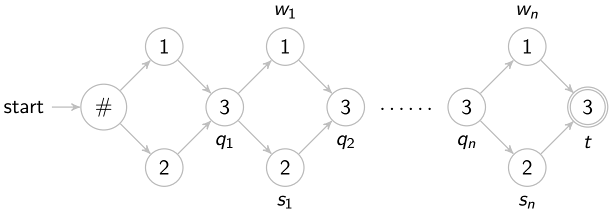
Theorem

If $\mathcal{L} = \mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}_w)$, with $\mathcal{L}$ Wheeler and $\mathcal{D}$ either minimum or Wheeler, then for all $q \in Q$, we have $\ell_q, r_q \in \mathbb{Q}$.

More on the algorithmic side on (Becker-Cenzato-Kim-Kodric-P.-Prezza. SPIRE 2023.)
More on the entanglement:



Minimum DFA vs. Minimum Wheeler-DFA



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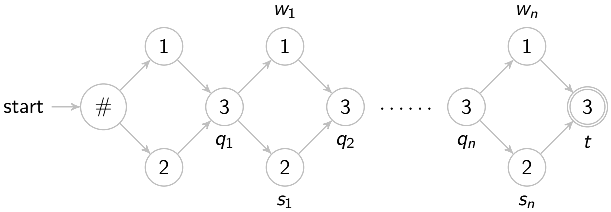

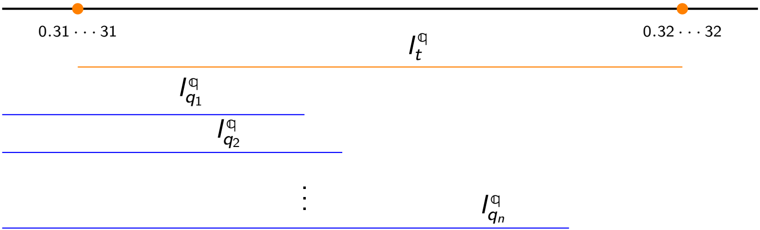

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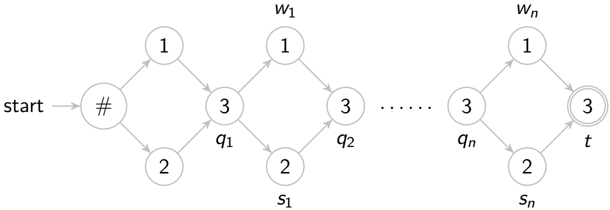
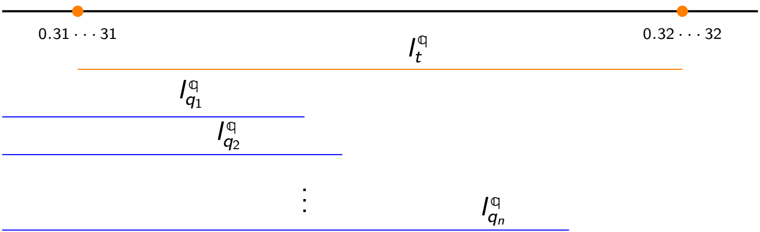
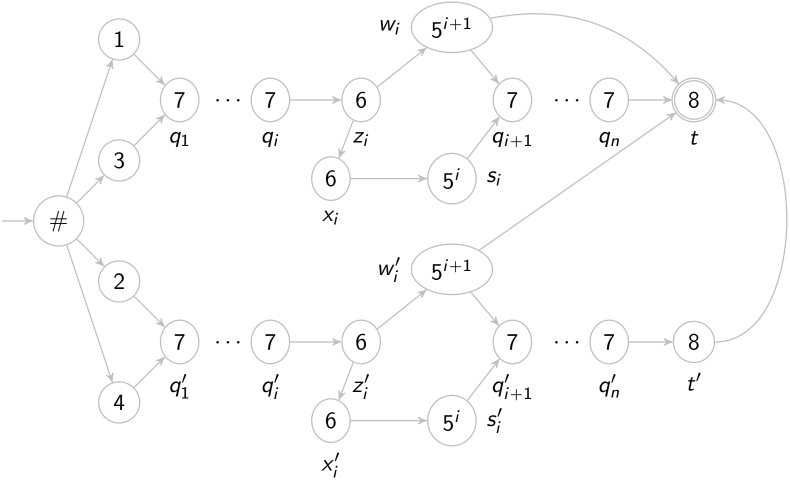
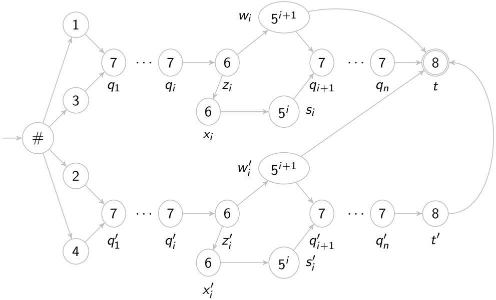




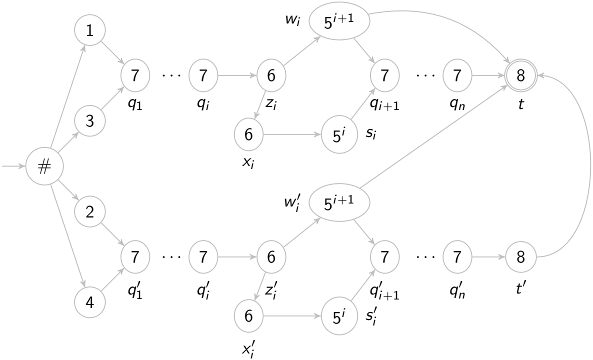
Question: is the $|Q_w|/|Q|$ related to width$(\mathcal{D})$?

Lower bound


1
3
#
2
4
7
7
6
6
w_i
5^{i+1}
5^i
s_i
7
7
8
q_1
q_i
z_i
x_i
q_{i+1}
q_n
t
w'_i
5^{i+1}
5^i
s'_i
q'_1
q'_i
z'_i
x'_i
q'_{i+1}
q'_n
t'
8



State type	Left limit	Right limit
$s_{i,j}$	$0.5^j6675^i67\ldots$	$0.5^j6675^{i-1}667\ldots$
$w_{i,j}$	$0.5^j675^i67\ldots$	$0.5^j675^{i-1}667\ldots$
x_i	$0.6675^i67\ldots$	$0.6675^{i-1}667\ldots$
z_i	$0.675^i67\ldots$	$0.675^{i-1}667\ldots$
q_i	$0.75^i67\ldots$	$0.75^{i-1}667\ldots$
t	$0.85^n67\ldots$	$0.8875^{n-1}667\ldots$
t'	$0.875^n67\ldots$	$0.875^{n-1}667\ldots$


#
1
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5^i
w_i
z_i
x_i
s_i
q_1
q_i
q_{i+1}
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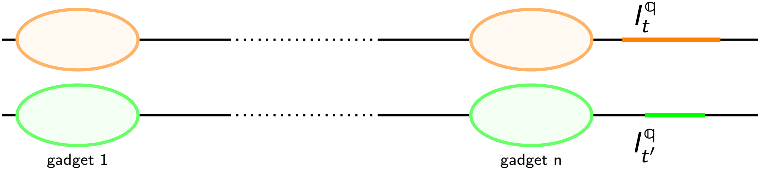
q'_{i+1}
q'_n
t'


$I_t^{\mathfrak{q}}$
$I_{t'}^{\mathfrak{q}}$
gadget 1
gadget n

Theorem

Let $\mathcal{L} = \mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}_w)$*, with* $\mathcal{L}$ *Wheeler,* $\mathcal{D}$ *minimum,* $\mathcal{D}_w$ *minimum Wheeler, and let* $f(\cdot,\cdot)$ *be such that* $|\mathcal{D}_w| = O(\, f(|\mathcal{D}|, width(D))\,)$*. Then, for any* $k, p \in \mathbb{N}$*,*

$$f(n, p) \notin O(n^k + 2^p).$$

The arithmetic way

Formally, for the *left* case, we consider the problem of finding the set of all real-valued vectors $x \in \mathbb{R}^Q$ that satisfy the following constraint satisfaction program, that we name $\mathcal{P}_{Left}$:

$$
\begin{aligned}
&(1) \quad x_s = 0, \\
&(2) \quad 0 < x_q < 1, && (\forall q \in Q \backslash \{s\}) \\
&(3) \quad (\sigma + 2) \cdot x_q - \lambda(q) = \min \left\{ x_{q'} \mid \delta(q') = q \right\}, && (\forall q \in Q \backslash \{s\})
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Lemma

Let $\mathcal{L}$ be a Wheeler language, and $\mathcal{D} = (Q, s, \delta, F)$ be either minimum or Wheeler accepting $\mathcal{L}$, and let $\ell \in \mathbb{Q}^Q$ be the vector of left limits. Then, ℓ is a solution of $\mathcal{P}_{Left}$.

Theorem

Let $\mathcal{D} = (Q, s, \delta, F)$ be either minimum or Wheeler accepting $\mathcal{L}$ Wheeler, and $\ell \in \mathbb{Q}^Q$ be the vector of left limits. Then, $\mathcal{P}_{Left}$ always admits ℓ as its unique solution.

The arithmetic way

Consider the following linear program $\mathcal{P}^*_{Left}$:

$$
\begin{array}{lll}
\text{maximize:} & \sum_{q \in Q} x_q, & \\
\text{subject to:} & x_s = 0, & \\
 & 0 < x_q < 1, & \forall q \in Q \backslash \{s\}, \\
 & (\sigma + 2) \cdot x_q - \lambda(q) \leqslant x_{q'}, & \forall q, q' \in Q \text{ s.t. } \delta(q') = q,
\end{array}
$$

Theorem

Let $\mathcal{D} = (Q, s, \delta, F)$ be either minimum or Wheeler accepting $\mathcal{L}$ Wheeler, and $\ell \in \mathbb{Q}^Q$ be the vector of left limits. Then $\mathcal{P}^*_{Left}$ always admits ℓ as its unique solution.

Conclusions and Open problems

- A parameter measuring the DFA vs. WDFA growth (*finite* entanglement)
- On-line splitting of minimum DFA (self-adjusting splitting)
- Optimal *disentanglement*

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Optimization of the Hasse automaton $\mathcal{H}$ construction

$$\text{width}(\mathcal{L}(\mathcal{H})) = \text{width}(\mathcal{H}) = \textit{ent}(\mathcal{H})$$

Thank you for your attention.