Computing the LCP Array of a Labeled Graph

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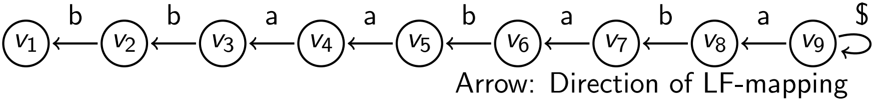
SA and LCP for $T = bbaababa\$$

j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	$

SA and LCP for $T = bbaababa\$$

j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	\$



SA and LCP for $T = bbaababa\$$

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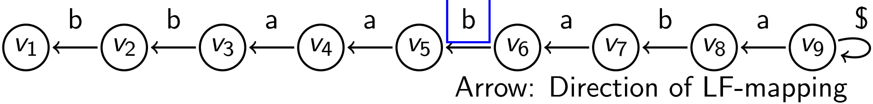
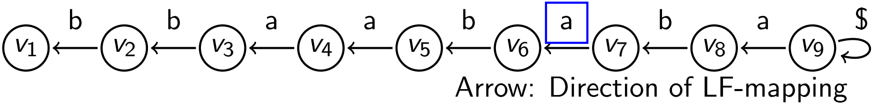
$T[j]$	b	b	a	a	b	a	b	a	\$



v_5: b

SA and LCP for $T = bbaababa\$$

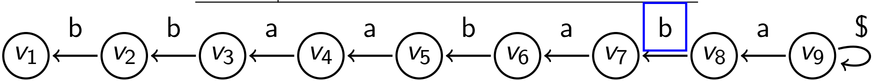
j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	\$



v_5: ba

SA and LCP for $T = bbaababa\$$

j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	\$

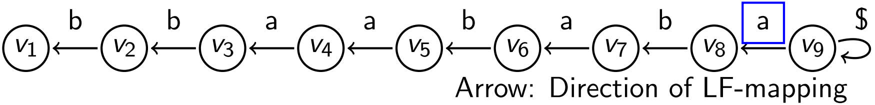


Arrow: Direction of LF-mapping

v_5: bab

SA and LCP for $T = bbaababa\$$

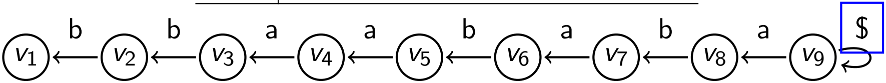
j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	\$



v_5: baba

SA and LCP for $T = bbaababa\$$

j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	$

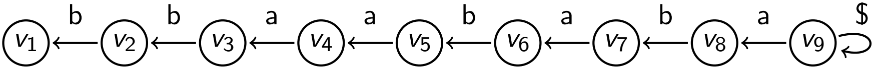


Arrow: Direction of LF-mapping

v_5: baba$

SA and LCP for $T = bbaababa\$$

j	1	2	3	4	5	6	7	8	9
$T[j]$	b	b	a	a	b	a	b	a	\$

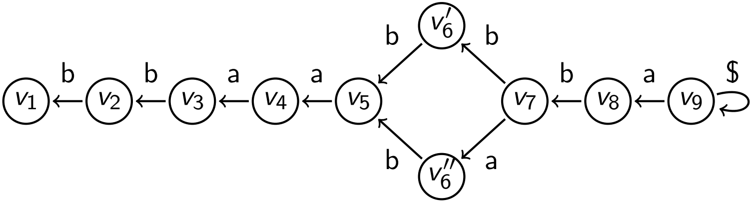


Arrow: Direction of LF-mapping

LCP	SA	Suffix
	9	\$
0	8	a\$
1	3	aababa\$
1	6	aba\$
3	4	ababa\$
0	7	ba\$
2	2	baababa\$
2	5	baba\$
1	1	bbaababa\$

General graph?

- There are possibly many strings associated with a node.



General graph?

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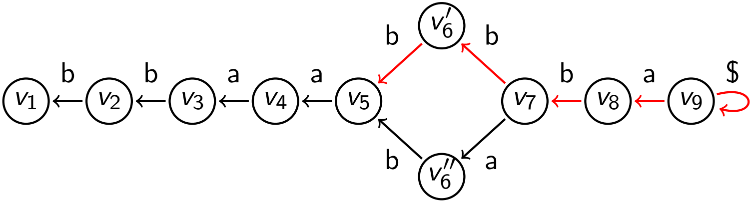
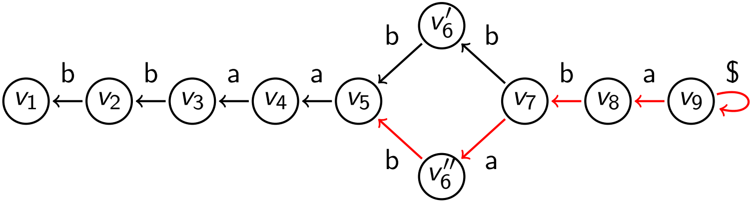


Strings associated with v_5: bbba$

General graph?

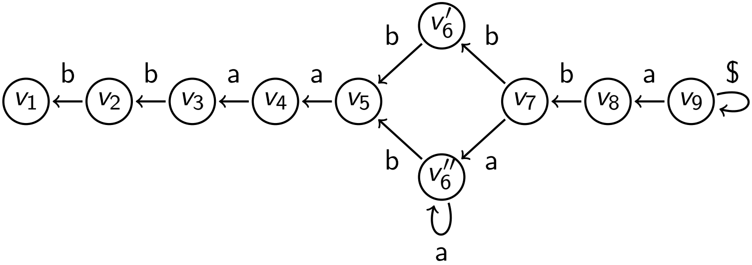
- There are possibly many strings associated with a node.



Strings associated with v_5: bbba$, baba$

General graph?

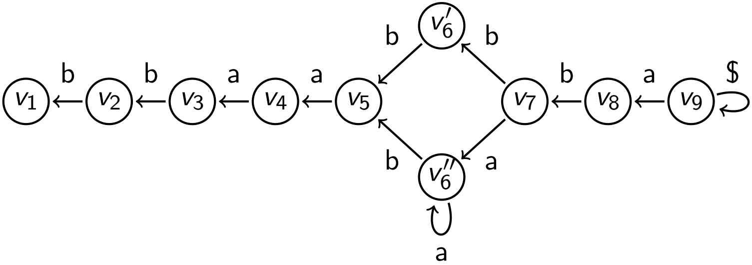
- There are possibly many strings associated with a node.



Strings associated with v_5: bbba\$, baba\$
baaba\$, baaaba\$, baaaaba\$, $\cdots$

General graph?

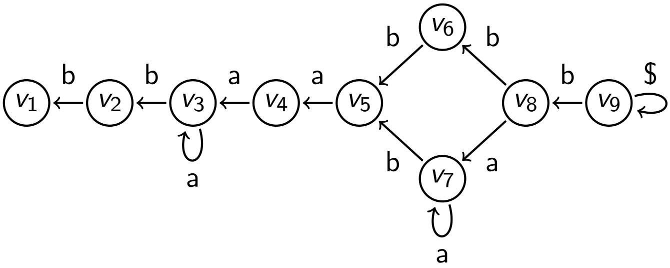
- There are possibly many strings associated with a node.



Strings associated with v_5: bbba\$, baba\$
baaba\$, baaaba\$, baaaaba\$, $\cdots$

- <u>Question: which strings should be considered?</u>
  - Answer: the smallest and the largest string.
    [Conte et al., DCC 2023], [Cotumaccio et al., SPIRE 2023]
  - e.g.) for v_5: baaaaa$\cdots$ and bbba\$

Infima and Suprema Strings



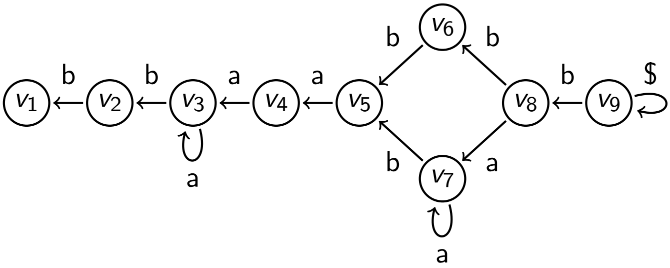
Node	inf	sup	Node	inf	sup
v_1	bbaaa$\cdots$	bbaabbb\$	v_6	bb\$	bb\$
v_2	baaa$\cdots$	baabbb\$	v_7	aaa$\cdots$	ab\$
v_3	aaa$\cdots$	aabbb\$	v_8	b\$	b\$
v_4	abaaa$\cdots$	abbb\$	v_9	\$	\$
v_5	baaa$\cdots$	bbb\$			

Motivation

- LCP array is useful for several problems related to string matching.
  - E.g., faster string matching, suffix tree functionalities.
- Recent work: Also useful for graph cases!
  - Navigation of de Bruijn graph [Boucher et al., DCC 2015]
  - Matching statistics [Conte et al., DCC 2023]

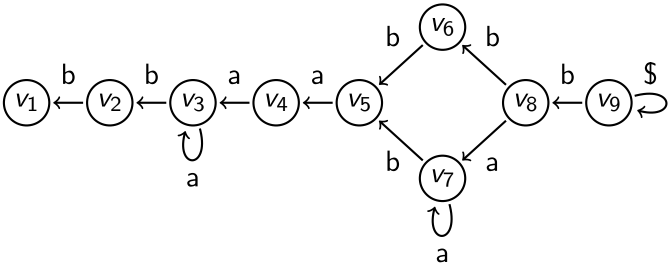
  - Computation algorithm for general graphs has not been proposed.

Infima and Suprema Strings



Node	inf	sup	Node	inf	sup
v_1	bbaaa···	bbaabbb$	v_6	bb$	bb$
v_2	baaa···	baabbb$	v_7	aaa···	ab$
v_3	aaa···	aabbb$	v_8	b$	b$
v_4	abaaa···	abbb$	v_9	$	$
v_5	baaa···	bbb$			

Infima and Suprema Strings



Node	inf	sup	Node	inf	sup
v_1	bbaaa··· (11)	bbaabbb$ (12)	v_6	bb$ (10)	bb$ (10)
v_2	baaa··· (8)	baabbb$ (9)	v_7	aaa··· (2)	ab$ (4)
v_3	aaa··· (2)	aabbb$ (3)	v_8	b$ (7)	b$ (7)
v_4	abaaa··· (5)	abbb$ (6)	v_9	$ (1)	$ (1)
v_5	baaa··· (8)	bbb$ (13)			

Sorted Infima and Suprema Strings

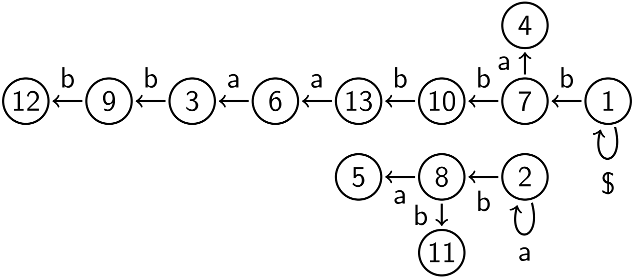
rank	string	nodes
1	\$	inf(9), sup(9)
2	aaa$\cdots$	inf(3), inf(7)
3	aabbb\$	sup(3)
4	ab\$	sup(7)
5	abaaa$\cdots$	inf(4)
6	abbb\$	sup(4)
7	b\$	inf(8), sup(8)
8	baaa$\cdots$	inf(2), inf(5)
9	baabbb\$	sup(2)
10	bb\$	inf(6), sup(6)
11	bbaaa$\cdots$	inf(1)
12	bbaabbb\$	sup(1)
13	bbb\$	sup(5)

LCP of Infima and Suprema Strings

rank	LCP	string	nodes
1		\$	inf(9), sup(9)
2	0	aaa···	inf(3), inf(7)
3	2	aabbb\$	sup(3)
4	1	ab\$	sup(7)
5	2	abaaa···	inf(4)
6	2	abbb\$	sup(4)
7	0	b\$	inf(8), sup(8)
8	1	baaa···	inf(2), inf(5)
9	3	baabbb\$	sup(2)
10	1	bb\$	inf(6), sup(6)
11	2	bbaaa···	inf(1)
12	4	bbaabbb\$	sup(1)
13	2	bbb\$	sup(5)

Graph encoding of Inf/sup strings

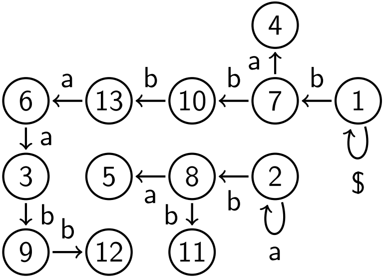
rank	string
1	\$
2	aaa···
3	aabbb\$
4	ab\$
5	abaaa···
6	abbb\$
7	b\$
8	baaa···
9	baabbb\$
10	bb\$
11	bbaaa···
12	bbaabbb\$
13	bbb\$



- Computed in $O(\min\{m \log n, n^2\})$ time.
  - Becker et al., ESA 2023
  - Cotumaccio, ISAAC 2023
  - Wheeler DFA: $O(m)$ time.
- Wheeler Pseudo-forest
  - Exactly one in-edge per node
  - Totally ordered (Wheeler graph)

LF-mapping works!

rank	string	out	
1	$	$	b
2	aaa· · ·	a	b
3	aabbb$		b
4	ab$		
5	abaaa· · ·		
6	abbb$	a	
7	b$	a	b
8	baaa· · ·	a	b
9	baabbb$		b
10	bb$		b
11	bbaaa· · ·		
12	bbaabbb$		
13	bbb$	a	

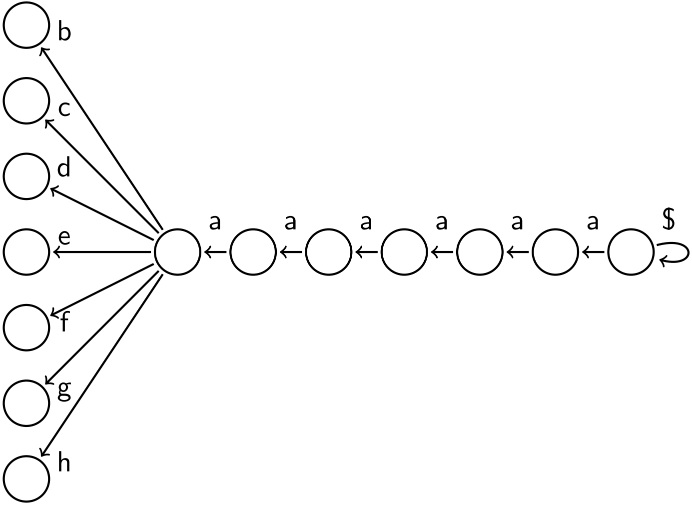


- Maybe LCP computation algorithm for strings work?

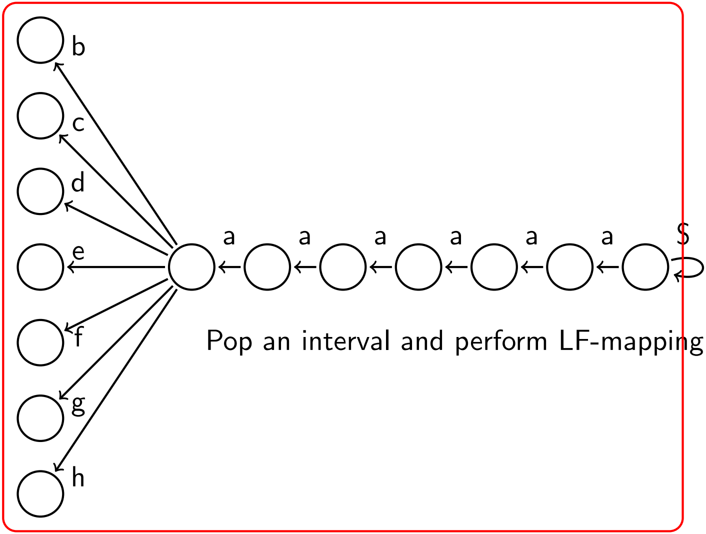
Main contribution

Algorithm	String	W. Pseudo-forest	Space (bits)
Manber and Myers	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$
Beller et al.	$O(n \log \sigma)$	$O(n\sigma \log n)$	$n \log \sigma + O(n)$
Ours	$O(n \log \sigma)$	$O(n \log \sigma)$	$O(n \log \sigma)$

Beller et al.'s Algorithm

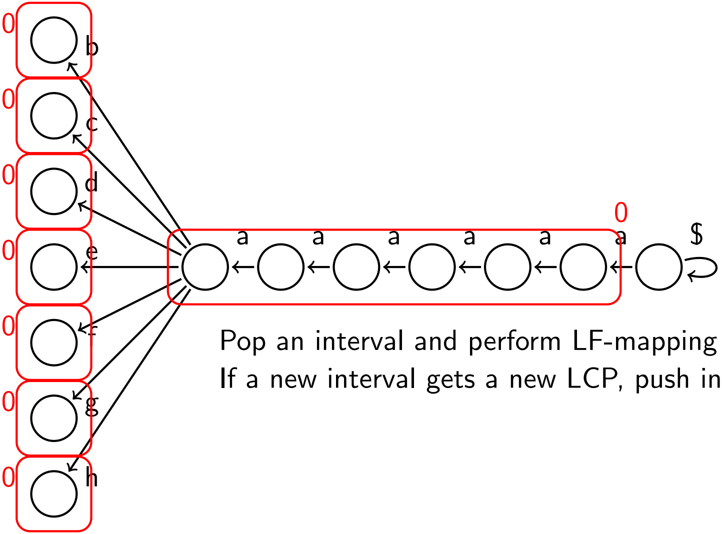


Beller et al.'s Algorithm



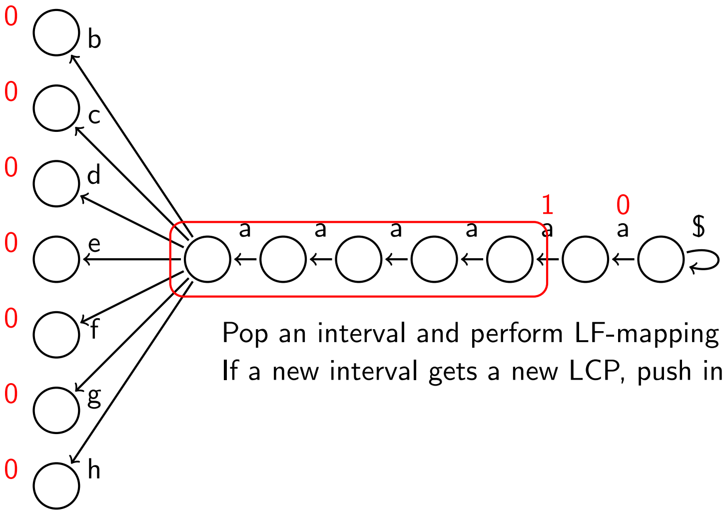
Pop an interval and perform LF-mapping for each char.

Beller et al.'s Algorithm



Pop an interval and perform LF-mapping for each char.
If a new interval gets a new LCP, push into the queue.

Beller et al.'s Algorithm



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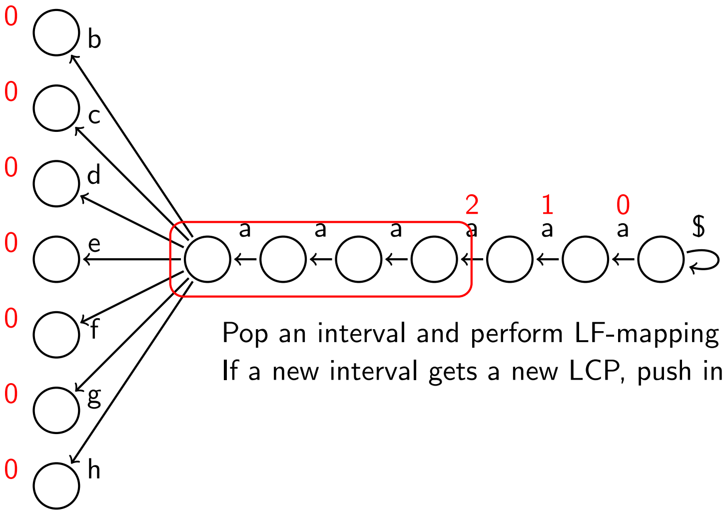

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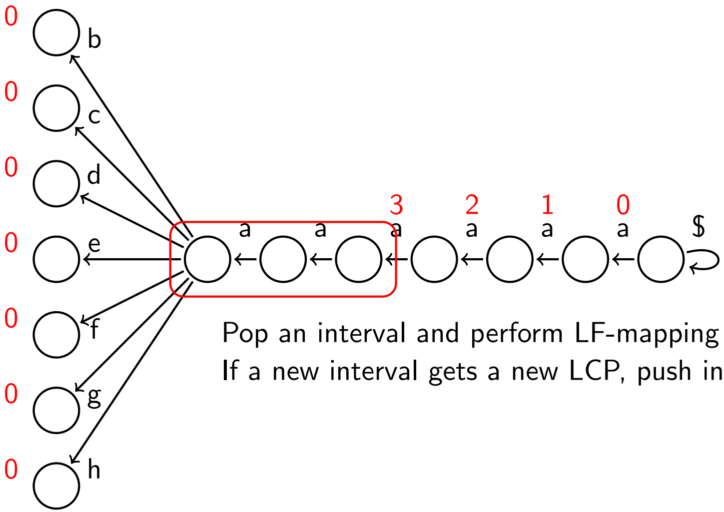

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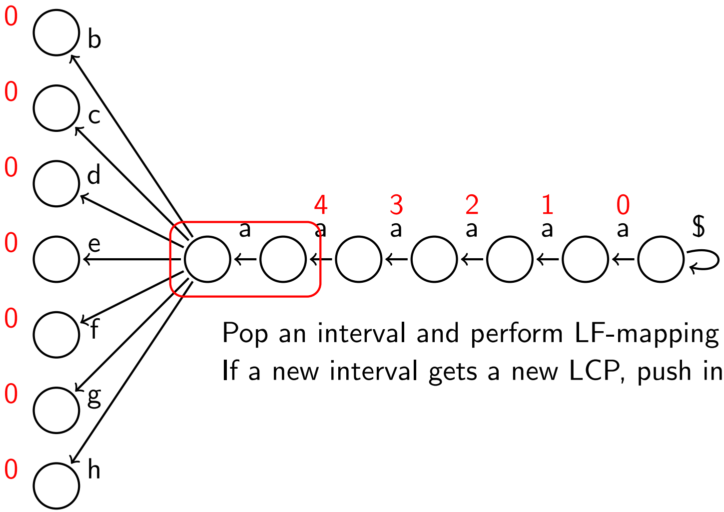

Pop an interval and perform LF-mapping for each char.
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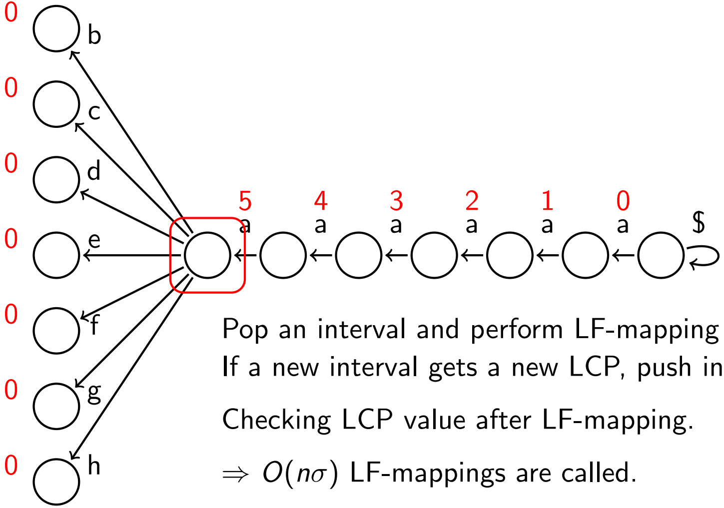
Beller et al.'s Algorithm



Pop an interval and perform LF-mapping for each char.
If a new interval gets a new LCP, push into the queue.

Checking LCP value after LF-mapping.

$\Rightarrow O(n\sigma)$ LF-mappings are called.

LCP Propagation

rank	string	out		LCP
1	$	$	b	0
2	aaa···	a	b	2
3	aabbb$		b	1
4	ab$			2
5	abaaa···			2
6	abbb$	a		0
7	b$	a	b	1
8	baaa···	a	b	3
9	baabbb$		b	1
10	bb$		b	2
11	bbaaa···			4
12	bbaabbb$			2
13	bbb$	a		

LCP Propagation

rank	string	out		LCP
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	2
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	3
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			4
13	bbb$	a		2

LCP Propagation

rank	string	out		LCP
1	\$	\$	b	
2	aaa· · ·	a	b	0
3	aabbb\$		b	2
4	ab\$			1
5	abaaa· · ·			2
6	abbb\$	a		2
7	b\$	a	b	0
8	baaa· · ·	a	b	1
9	baabbb\$		b	3
10	bb\$		b	1
11	bbaaa· · ·			2
12	bbaabbb\$			4
13	bbb\$	a		2

LCP Propagation

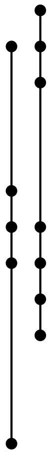
rank	string	out		LCP
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	2
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	3
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			4
13	bbb$	a		2

LCP Propagation

rank	string	out		LCP
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	2
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	3
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			4
13	bbb$	a		2

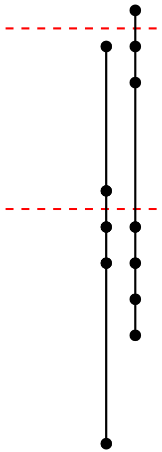
LCP Propagation with Interval Stabbing

rank	string	out			lcp
1	\$	\$		b	
2	aaa···		a	b	
3	aabbb\$			b	
4	ab\$				
5	abaaa···				
6	abbb\$		a		
7	b\$		a	b	
8	baaa···		a	b	
9	baabbb\$			b	
10	bb\$			b	
11	bbaaa···				
12	bbaabbb\$				
13	bbb\$		a		



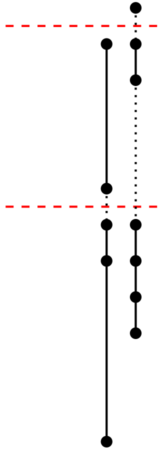
LCP Propagation with Interval Stabbing

rank	string	out	lcp
1	$	$ b	
2	aaa···	a b	0
3	aabbb$	b	
4	ab$		
5	abaaa···		
6	abbb$	a	
7	b$	a b	0
8	baaa···	a b	
9	baabbb$	b	
10	bb$	b	
11	bbaaa···		
12	bbaabbb$		
13	bbb$	a	



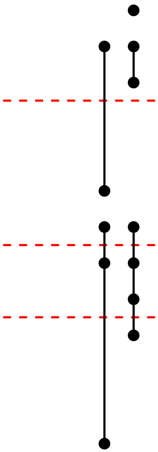
LCP Propagation with Interval Stabbing

rank	string	out		lcp
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	
4	ab$			1
5	abaaa···			
6	abbb$	a		
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	
10	bb$		b	1
11	bbaaa···			
12	bbaabbb$			
13	bbb$	a		



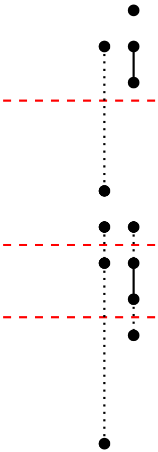
LCP Propagation with Interval Stabbing

rank	string	out	lcp
1	$	$ b	0
2	aaa···	a b	
3	aabbb$	b	1
4	ab$		
5	abaaa···		
6	abbb$	a	0
7	b$	a b	1
8	baaa···	a b	
9	baabbb$	b	1
10	bb$	b	
11	bbaaa···		
12	bbaabbb$		
13	bbb$	a	



LCP Propagation with Interval Stabbing

rank	string	out		lcp
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	2
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			
13	bbb$	a		2



LCP Propagation with Interval Stabbing

rank	string	out		lcp
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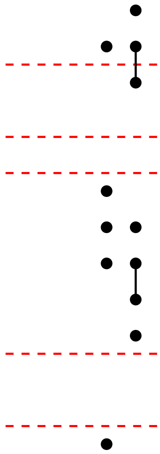
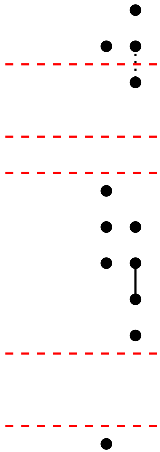
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			
13	bbb$	a		2



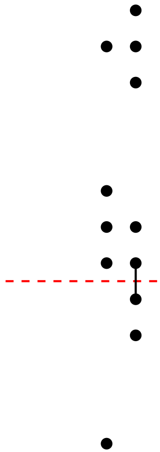
LCP Propagation with Interval Stabbing

rank	string	out	lcp
1	$	$ b	
2	aaa···	a b	0
3	aabbb$	b	2
4	ab$		1
5	abaaa···		2
6	abbb$	a	2
7	b$	a b	0
8	baaa···	a b	1
9	baabbb$	b	3
10	bb$	b	1
11	bbaaa···		2
12	bbaabbb$		
13	bbb$	a	2



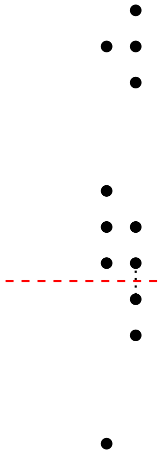
LCP Propagation with Interval Stabbing

rank	string	out		lcp
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	2
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	3
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			
13	bbb$	a		2



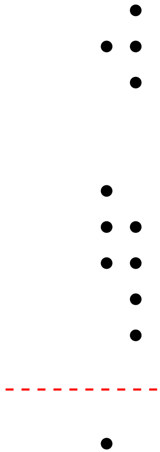
LCP Propagation with Interval Stabbing

rank	string	out		lcp
1	$	$	b	
2	aaa···	a	b	0
3	aabbb$		b	2
4	ab$			1
5	abaaa···			2
6	abbb$	a		2
7	b$	a	b	0
8	baaa···	a	b	1
9	baabbb$		b	3
10	bb$		b	1
11	bbaaa···			2
12	bbaabbb$			4
13	bbb$	a		2

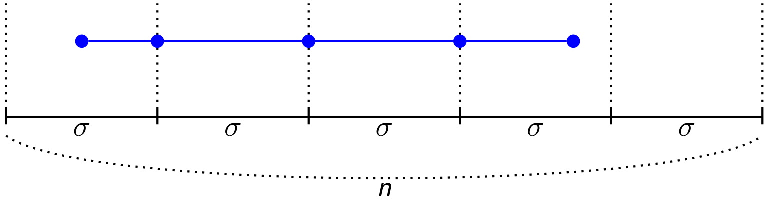


LCP Propagation with Interval Stabbing

rank	string	out	lcp
1	$	$ b	
2	aaa···	a b	0
3	aabbb$	b	2
4	ab$		1
5	abaaa···		2
6	abbb$	a	2
7	b$	a b	0
8	baaa···	a b	1
9	baabbb$	b	3
10	bb$	b	1
11	bbaaa···		2
12	bbaabbb$		4
13	bbb$	a	2



Partitioned Interval Tree



- Split the universe into n/σ blocks.
- Cut the intervals crossing block boundaries.
- Each block is indexed with an interval tree.

Partitioned Interval Tree

- The total number of (segmented) intervals: $O(n)$
  - Intervals contained in boundaries: $O(n)$
  - Segments of intervals cut by boundaries: $O(\frac{n}{\sigma} \cdot \sigma) = O(n)$.
- Each word in an interval tree uses $O(\log \sigma)$ bits.
  - In each block, there are at most σ^2 intervals.
  - Block size is σ.
- Total space: $O(n \log \sigma)$ bits

- Time for each stab-and-remove query: $O(\log \sigma + k)$ where k is the number of segments to be removed.
- Total time amortizes to $O(n \log \sigma)$.

- Construction (line sweeping): $O(n \log \sigma)$ time and space (bits).

Conclusions

Algorithm	String	W. Pseudo-forest	Space (bits)
Manber and Myers	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$
Beller et al.	$O(n \log \sigma)$	$O(n\sigma \log n)$	$n \log \sigma + O(n)$
Ours	$O(n \log \sigma)$	$O(n \log \sigma)$	$O(n \log \sigma)$

- Future Work:
  - Efficient computation of inf/sup graphs: $O(m)$ time?
  - $O(n)$ time or $n \log \sigma$ bits space?

Thank You!

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