Simplified Tight Bounds for Monotone Minimal Perfect Hashing

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Content

- Monotone minimal perfect hash function
- Known upper and lower bounds
- Model and counting argument
- Let's count

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- $f(x) = k$ if $x = a_k$ for some $k \in [1..n]$
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Example

$$u = 16, \{a_1, ..., a_5\} = \{3, 6, 7, 10, 14\}$$

1 2 <u>3</u> 4 5 <u>6</u> <u>7</u> 8 9 <u>10</u> 11 12 13 <u>14</u> 15 16

MMPHF colors the segment $[1..u]$: the color of x is $f(x)$

color 1, color 2, color 3, color 4, color 5

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Tight upper bound: $O(n \log\log\log \frac{u}{n})$
For all reasonable $n \le u < (1+\epsilon)n$, known facts give tight bounds

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graph of data structures,
chromatic number,
fractional chromatic number,
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Simpler?

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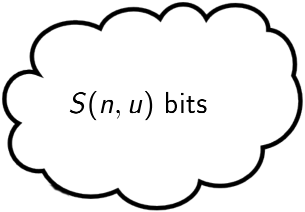

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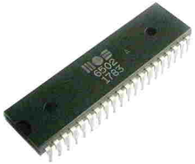

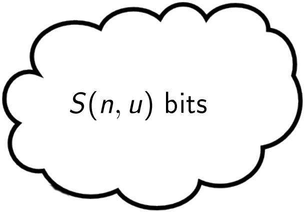

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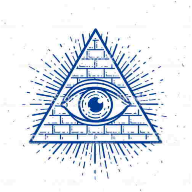

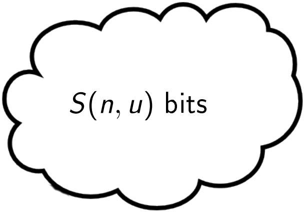

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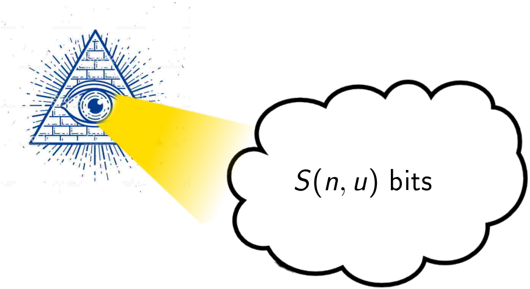

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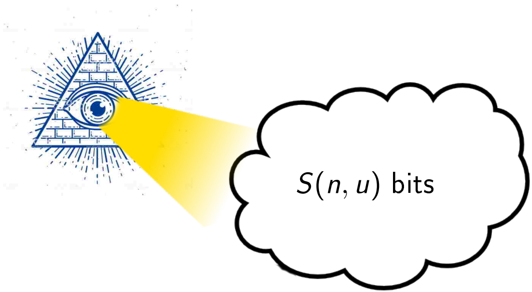

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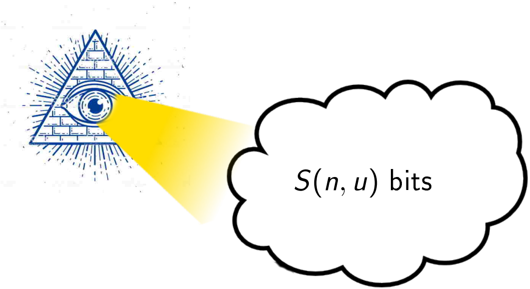

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Modelled as a function $f : [1..u] \times \{0, 1\}^S \to [1..n]$

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$\underline{1}$ 2 3 4 5 $\underline{6}$ 7 8 9 10 $\underline{11}$ 12 13 14 $\underline{15}$ $\underline{16}$

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A coloring is a map $f\colon [1..u] \to [1..n]$; suppose that $\mathcal{C}$ is a minimal family of colorings such that every tuple $\{a_1, \ldots, a_n\} \subset [1..u]$ is encoded by some $f \in \mathcal{C}$, i.e., $f(a_k) = k$ for all $k \in [1..n]$

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Claim

The MMPHF requires $S(n, u) \geq \log |\mathcal{C}|$ bits of space, so we have to prove that $\log |\mathcal{C}| \geq \Omega(n \log\log\log \frac{u}{n})$ for $u \geq (1 + \epsilon)n$

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  Since $n \log\log\log \frac{u}{n} = \Theta(n)$, the lower bound $\Omega(n)$ follows from the known bound $\Omega(n)$

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Split $[1..u]$ into $n/\bar{n}$ blocks of size $\bar{u} = u/(n/\bar{n})$, where $\bar{n} = \lfloor (\log\log \frac{u}{n})^{1/3} \rfloor$

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Then $\log|\mathcal{C}| \ge \log(\bar{n}^{\Omega(n)}) = \Omega(n \log \bar{n}) = \Omega(n \log\log\log \frac{u}{n})$

Omitted: upper and lower bounds for $n \leq u < (1+\epsilon)n$, upper bound $O(n \log\log\log \frac{u}{n})$ for $u \geq (1+\epsilon)n$, randomized MMPHF, the random process of Assadi et al. for $u = 2^{2^{n^3}}$

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