Subsequences with Generalised Gap Constraints

Upper and Lower Complexity Bounds

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$$p = \quad \mathtt{x} \quad \mathtt{y} \quad \mathtt{z} \quad \mathtt{y} \quad \mathtt{x}$$

$$w = \quad \mathtt{z} \quad \mathbf{x} \quad \mathbf{y} \quad \mathtt{x} \quad \mathbf{z} \quad \mathtt{x} \quad \mathbf{y} \quad \mathtt{y} \quad \mathbf{x}$$

$w =$	z	**x**	**y**	x	**z**	x	**y**	y	**x**
		$\uparrow$	$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$
		$e(1)$	$e(2)$		$e(3)$		$e(4)$		$e(5)$

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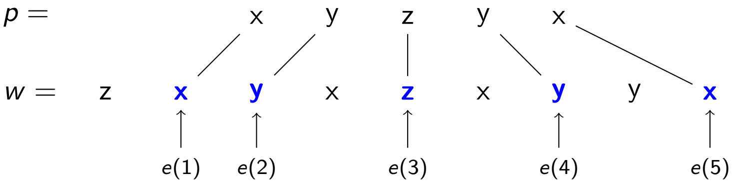

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 & & & & & & & & & \\
w = & \mathrm{z} & \mathbf{x} & \mathbf{y} & \mathrm{x} & \mathbf{z} & \mathrm{x} & \mathbf{y} & \mathrm{y} & \mathbf{x} \\
 & & \uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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\end{array}$$

Notation: $p \preceq_e w$ with $e : [m] \to [n]$ (or just $p \preceq w$)

Properties of e:

- $e(1) < e(2) < \cdots < e(m)$
- $w[e(i)] = p[i]$

Gaps in subsequences

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Definition

Given positions $i, j \in [m], i < j$, define

$$\mathrm{gap}_{w,e}[i,j] := w[e(i)+1..e(j)-1]$$

as the gap *between* the embedding of the i-th and j-th character.

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$$\mathrm{gap}_{w,e}[2,4] = \mathrm{xzx}$$

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Example:

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$$\mathrm{gap}_{w,e}[1,2] = \varepsilon$$

Gap-Constraints

Definition

A gap constraint is defined as a triple $C = (i, j, L)$ with $i, j \in [m], i < j$ and $L \subseteq \Sigma^*$. An embedding e is said to satisfy the constraint if and only if

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Semilinear sets

Definition

A subset $L \subseteq \mathbb{N}$ is called *linear*, if it is of the form

$$L = L(x_0; x_1, \ldots, x_m) = \{x_0 + \sum_{i=1}^{m} k_i x_i \mid k_1, \ldots, k_m \in \mathbb{N}_0\}$$

with $x_0 \in \mathbb{N}_0, x_1, \ldots, x_k \in \mathbb{N}$. A *semilinear* set is a finite union of linear sets.

Size of a (semi-)linear set:

- A linear set $L = L(x_0; x_1, \ldots, x_m)$ has size $\text{size}(L) = m + 1$.
- A semilinear set $L = \bigcup_{i=1}^{k} L_i$ with L_i linear sets has size $\text{size}(L) = \sum_{i=1}^{k} \text{size}(L_i)$.

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We use $\textsc{Match}_{\mathsf{REG}}$ and $\textsc{Match}_{\mathsf{SLS}}$ to denote the variants of the matching problem with regular and semilinear length constraints respectively.

Polynomial solutions for $\textsc{Match}$ with constant $|\mathcal{C}|$

$\textsc{Match}_{\mathsf{REG}}$: Construct an NFA to solve the problem:

Theorem

$\textsc{Match}_{\mathsf{REG}}$ *can be solved in polynomial time for constant* $|\mathcal{C}|$ *and is fixed parameter tractable for the combined parameter* $(|p|, \mathsf{gapsize}(\mathcal{C}))$.

$\textsc{Match}_{\mathsf{SLS}}$: Enumerate all possible partial embeddings of the positions that have a constraint, then "fill the gaps":

Theorem

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Hardness of $\textsc{Match}_{\mathsf{SLS}}$

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Example:

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

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Theorem

$\textsc{Match}_{\mathsf{SLS}}$ *parameterised by* $|p|$ *is* $W[1]$*-hard, even for constant* gapsize$(\mathcal{C})$ *and binary alphabet* Σ*.*

Hardness of $\textsc{Match}_{\mathsf{REG}}$

Definition (1-in-3-3SAT)

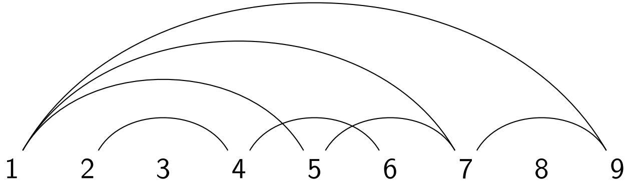
Given a set of variables $A = \{x_1, \ldots, x_n\}$ and clauses $c_1, \ldots, c_m \subseteq A$ with $|c_j| = 3$, find a subset $B \subseteq A$, such that $|c_i \cap B| = 1$ for every $j \in [m]$.

- Pattern $p = (\mathtt{b}\#)^n(\mathtt{b}\#)^m$
- Text $w = (\mathtt{bb}\#)^n(\mathtt{bbb}\#)^m$
- First part: decide $x_i \in B$ for $i \in [n]$.
- Second part: decide which of the 3 variables from c_j is in B for $j \in [m]$ (need some ordering on the variables).
- Use constraints to ensure that assignments are consistent.

Theorem

$\textsc{Match}_{\mathsf{REG}}$ is NP-complete, even for binary alphabet Σ and with gap-constraints that can be represented by DFAs with at most 8 states.

The Graph Structure of Constraints



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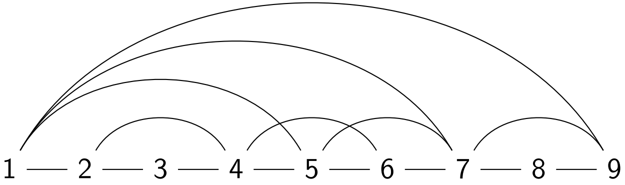

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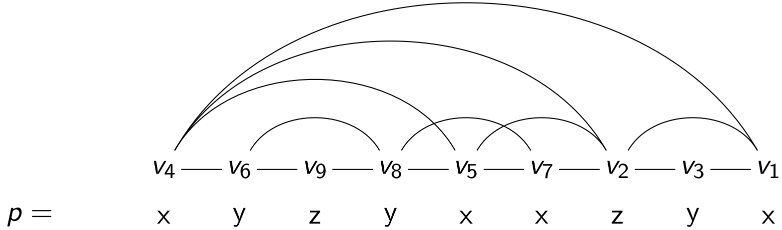

- Construct partial embeddings by adding the positions one-by-one in order $\sigma = (v_1, \ldots, v_m)$.
- At each time t, we compute all possible partial embeddings, but we only care about the the values $e(v_i)$, where $i \leq t$ and $(v_i, v_j) \in E$ for some $j > t$.
- When adding $e(v_t)$ to the embedding, combine all previously possible embeddings with all possible values of $e(v_t)$, then remove all embeddings that violate any constraint involving v_t.

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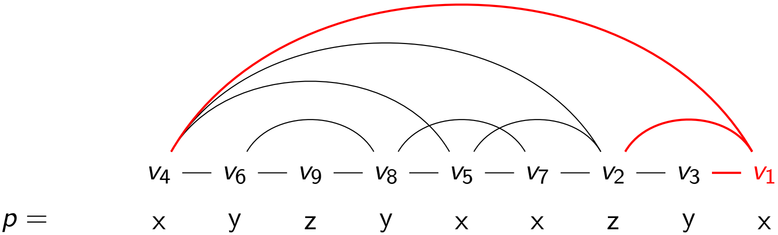

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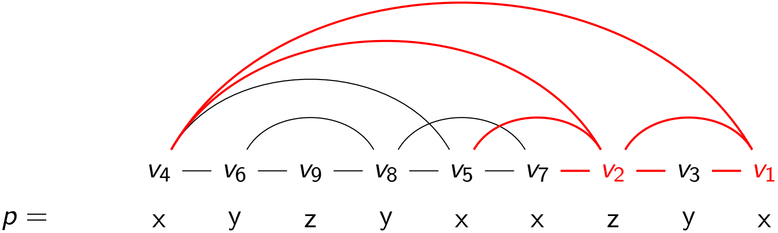

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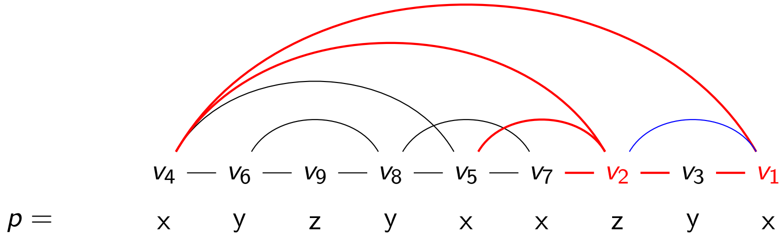

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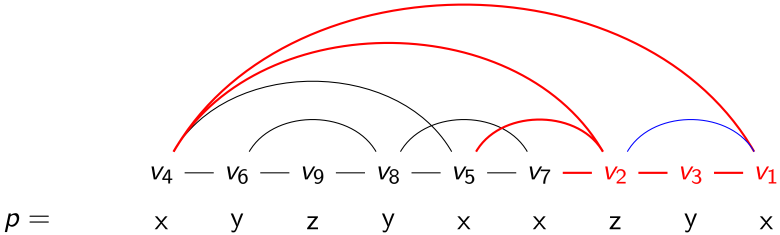

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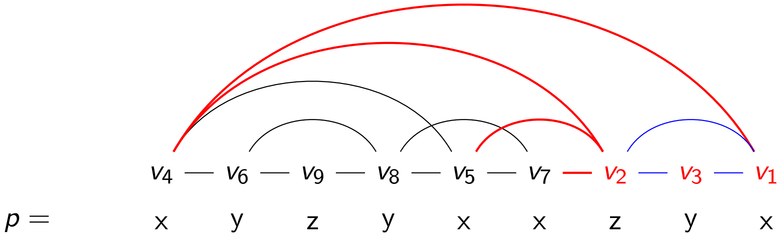

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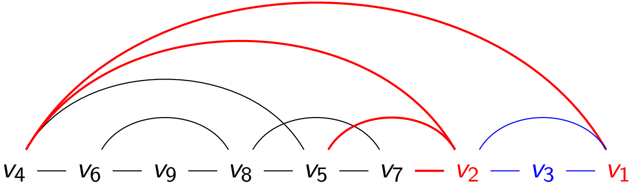

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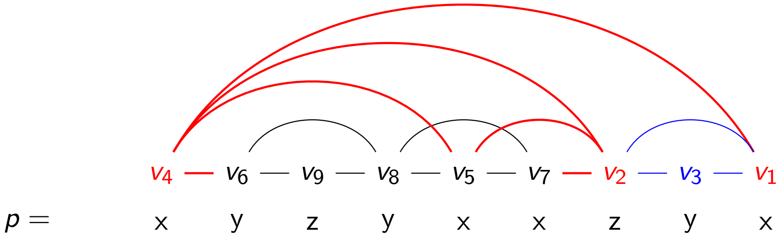

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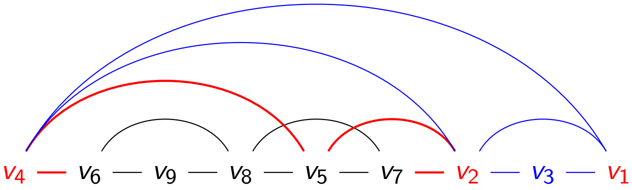
The Graph Structure of Constraints



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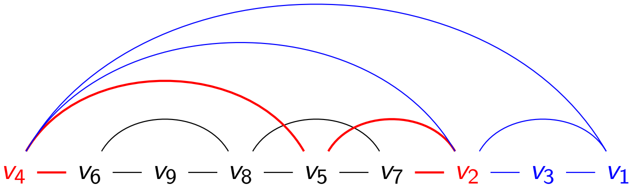

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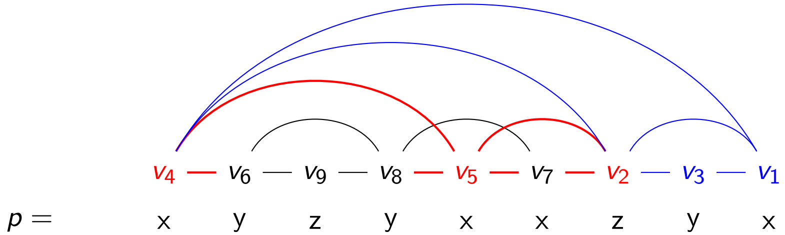

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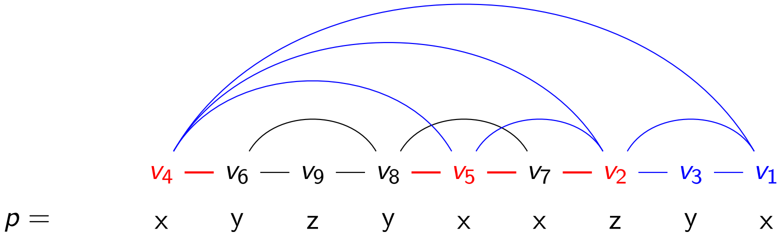

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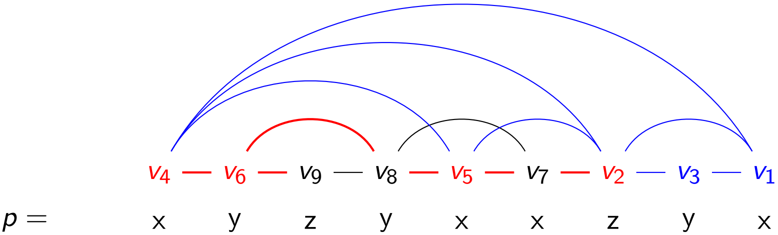

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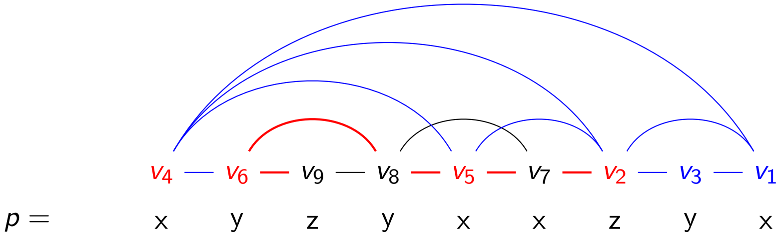

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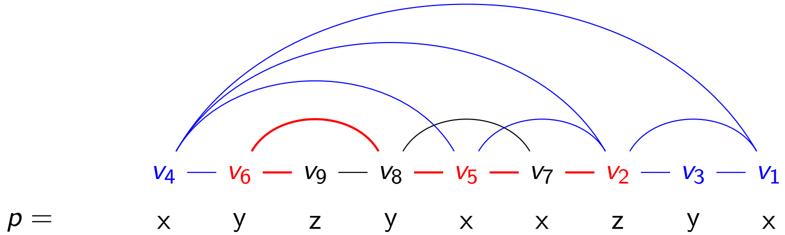

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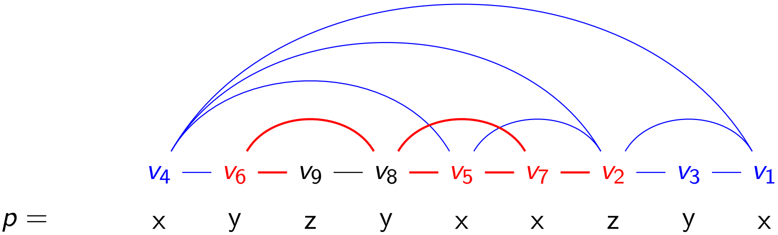

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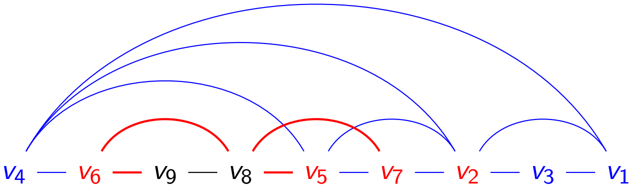

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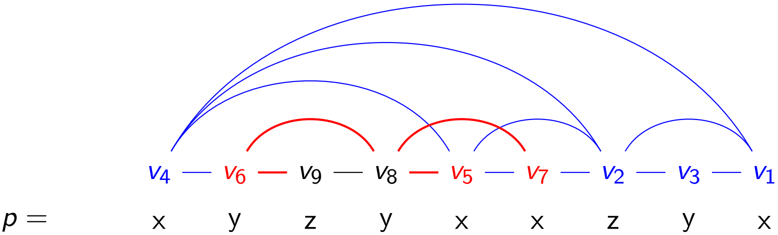

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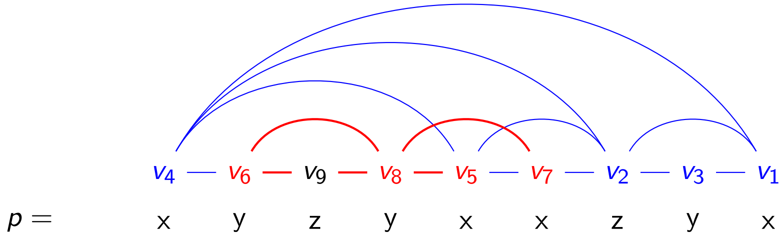

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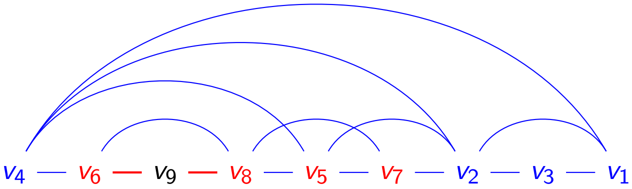

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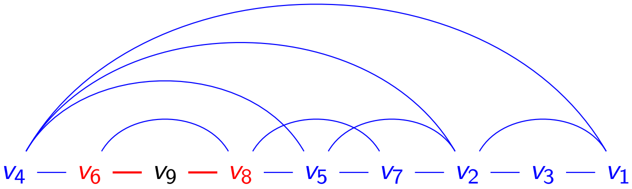

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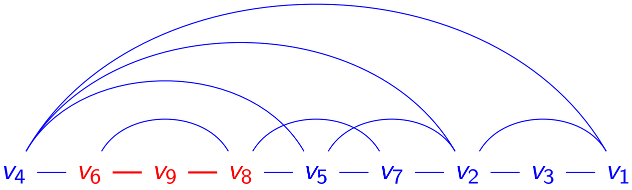

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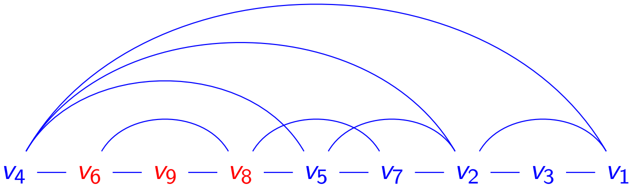

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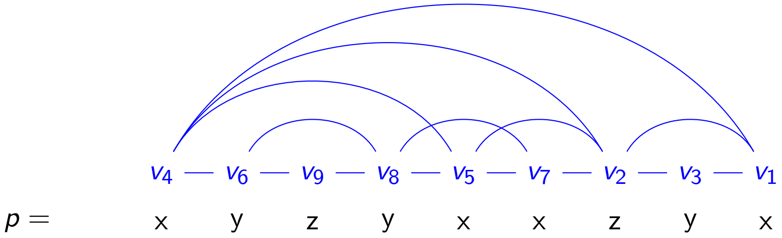

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The Graph Structure of Constraints

Definition (Vertex separation number)

Given a linear ordering $\sigma = (v_1, \ldots, v_n)$ of vertices in a graph, the vertex separation number of σ is the smallest number s such that, for each vertex v_i at most s vertices of $v_1, \ldots, v_{i-1}$ have some v_j with $j \geq i$ as neighbour. The vertex separation number of a graph is the minimum vertex separation number over all linear orderings of the graph.

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Theorem

If the vertex separation number of the constraint graph is bound by k, $\textsc{Match}_{\mathsf{REG}}$ and $\textsc{Match}_{\mathsf{SLS}}$ can be solved in $\mathcal{O}(m^2 n^{k+1} + m^2 n^2 \log\log n)$ and $\mathcal{O}(m^2 n^{k+1})$ time respectively.

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The second part is witnessed by the k-$\textsc{Clique}$ reduction from earlier.

The Interval Structure of Constraints

For $C = (i, j, L)$ define *interval*$(C) = [i, j - 1]$.

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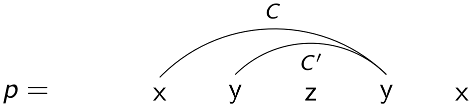

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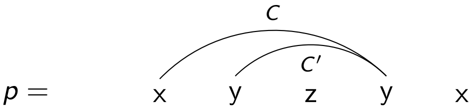

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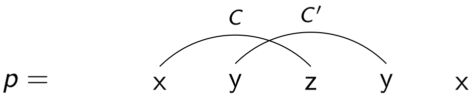

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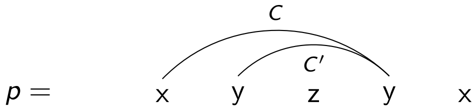

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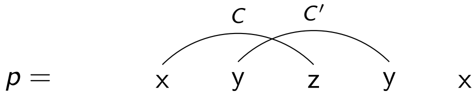
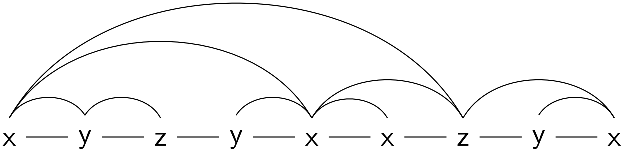


What about $\textsc{Match}$ with pairwise non-intersecting constraints?

Matching with non-intersecting constraints

$p =$



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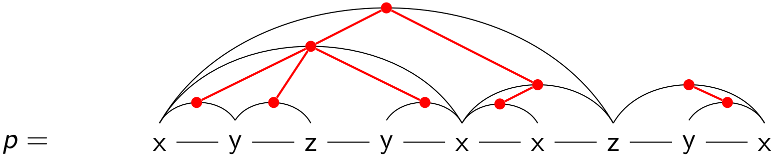

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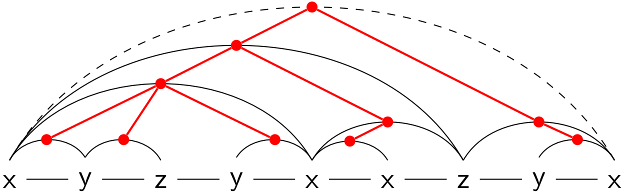

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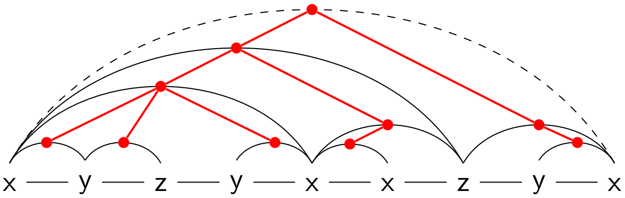
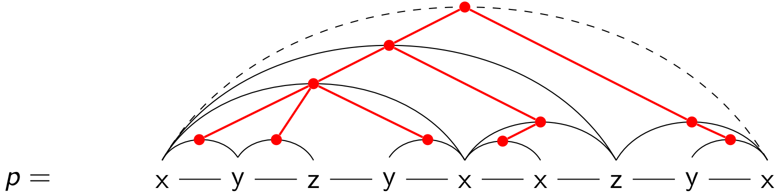


We can recursively construct the partial embeddings of $p[i..j]$ for each constraint (i, j, L) bottom-up the tree.

Matching with non-intersecting constraints



We can recursively construct the partial embeddings of $p[i..j]$ for each constraint (i, j, L) bottom-up the tree.

Theorem

If constraints are pairwise non-intersecting, $\textsc{Match}_{\mathsf{REG}}$ *and* $\textsc{Match}_{\mathsf{SLS}}$ *can be solved in* $\mathcal{O}(n^\omega k + n^2 k \log\log n)$ *and* $\mathcal{O}(n^\omega k)$ *time respectively, where* $\mathcal{O}(n^\omega)$ *is the time needed to multiply two boolean matrices of size* $n \times n$.

Matching with non-intersecting constraints

Definition (3-OV)

Given three sets $A = \{\vec{a}_1, \ldots, \vec{a}_n\}, B = \{\vec{b}_1, \ldots, \vec{b}_n\}, C = \{\vec{c}_1, \ldots, \vec{c}_n\}$ of d-dimensional boolean vectors, are there indices $i, j, k \in [n]$, such that $\vec{a}_i \cdot \vec{b}_j \cdot \vec{c}_k = \vec{0}$?

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$$p = \overline{\mathsf{C}}_p(\vec{a}_n) \ \ldots \ \overline{\mathsf{C}}_p(\vec{a}_i) \ \ldots \ \overline{\mathsf{C}}_p(\vec{a}_1) \quad \S \quad \mathsf{C}_p(\vec{a}_1) \ \ldots \ \mathsf{C}_p(\vec{a}_i) \ \ldots \ \mathsf{C}_p(\vec{a}_n)$$

$$w = \quad w_0 \ \overline{\mathsf{C}}_w(\vec{b}_n) \ldots \ \overline{\mathsf{C}}_w(\vec{b}_1) \ w_0 \quad \S \quad w_0 \ \mathsf{C}_w(\vec{c}_1) \ldots \ \mathsf{C}_w(\vec{c}_n) \ w_0$$

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Theorem

Both variants of MATCH *with pairwise non-intersecting constraints cannot be solved in $\mathcal{O}(n^g k^h)$ time with $g + h < 3$, unless the Strong Exponential Time Hypothesis fails.*

Thank you for your attention!

Summary:

- *Problem:* Matching subsequences with gap constraints
- Two types of constraints: regular and semilinear length constraints
- Polynomial solutions for constant amount of constraints
- Hardness of the problem parameterized by the length of the pattern, witnessed by reductions from k-$\textsc{Clique}$ and 1-in-3-3-SAT.
- Graph structure of Constraints: Relation between complexity of the problem and vertex separation number of constraint graph
- Interval structure of Constraints: Efficient solution for the case of non-intersecting constraints, lower bound via fine-grained reduction from 3-OV.

Do you have any questions?