Efficient Construction of Long Orientable Sequences

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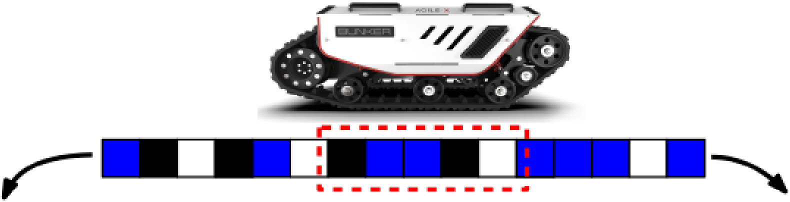
A scenario

- Consider an autonomous robot with limited sensors on a cyclic track.



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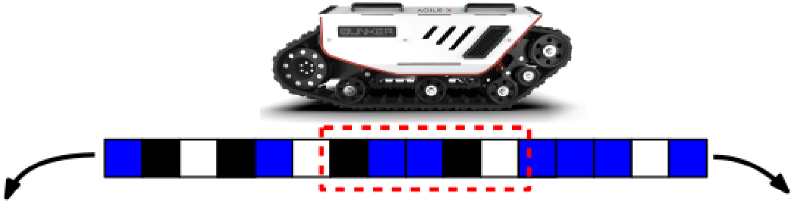

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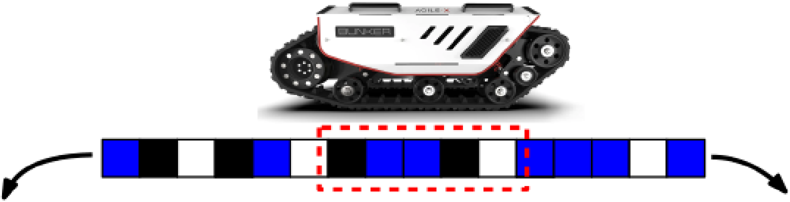

- To determine its location on the track, labelled with a finite set of colored squares, the robot scans a window of *n* squares directly beneath it.
- Suppose the track is labelled with only white and black squares.
- What constraints on the sequence of white and black squares allow the robot to uniquely determine its position *and* orientation?

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Definition 1

We say that a binary circular string w is an *orientable sequence of order n* (or an $\mathcal{OS}(n)$) if every length-n string occurs at most once in w, and if a length-n string u occurs in w, then u^R is not a substring of w.

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Example 2

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Background

- Dai, Martin, Robshaw, and Wild [Cryptography and Coding III (1993)] first introduced orientable sequences.

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- They show the existence of an $\mathcal{OS}(n)$ with asymptotically optimal length.
  - Did not give a constructive proof.
  - Left the problem of efficiently generating such a sequence open.

Background

Dai et al. showed that L_n is the following, where μ is the Möbius function:

$$L_n = \left(2^{n-1} - \frac{1}{2}\sum_{d|n}\mu(n/d)\frac{n}{d}H(d)\right), \quad \text{where} \quad H(d) = \frac{1}{2}\sum_{i|d} i\left(2^{\lfloor \frac{i+1}{2} \rfloor} + 2^{\lfloor \frac{i}{2} \rfloor + 1}\right).$$

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Their upper bound U_n is the following:

$$U_n = \begin{cases} 2^{n-1} - \frac{41}{9}2^{\frac{n}{2}-1} + \frac{n}{3} + \frac{16}{9} & \text{if } n \bmod 4 = 0, \\ 2^{n-1} - \frac{31}{9}2^{\frac{n-1}{2}} + \frac{n}{3} + \frac{19}{9} & \text{if } n \bmod 4 = 1, \\ 2^{n-1} - \frac{41}{9}2^{\frac{n}{2}-1} + \frac{n}{6} + \frac{20}{9} & \text{if } n \bmod 4 = 2, \\ 2^{n-1} - \frac{31}{9}2^{\frac{n-1}{2}} + \frac{n}{6} + \frac{43}{18} & \text{if } n \bmod 4 = 3. \end{cases}$$

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- The length of their constructed $\mathcal{OS}(n)$ is not of asymptotically optimal length.
- Their construction requires storing the whole string.

Main results: a successor rule

- Let Σ be some finite nonempty alphabet.
- Let w be a circular string over Σ such that for some $n \geq 1$, every length-n substring of w occurs exactly once.

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- For example, the function $g : \{0,1\}^3 \to \{0,1\}$ defined by $g(\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3) = (\mathrm{a}_1 + 1) \bmod 2$ is a successor rule for 000111.

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  - Space complexity is logarithmic in L_n.
  - Time complexity (per bit) is logarithmic in L_n.

Strategy

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- Use cycle-joining to join all these “short” orientable sequences into one large orientable sequence of length L_n.

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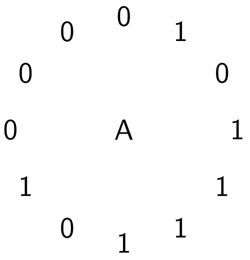
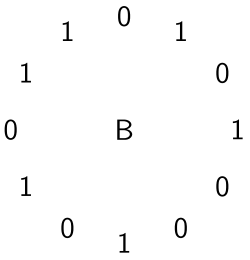
- Let a, b, c, and d be symbols such that aub is a substring of α and cud is a substring of β.
- Then α and β can be cycle-joined by exchanging the successor of au with the successor of cu, so that aud and cub are substrings of the cycle-joined result.

Cycle-joining example

Let $n = 6$.

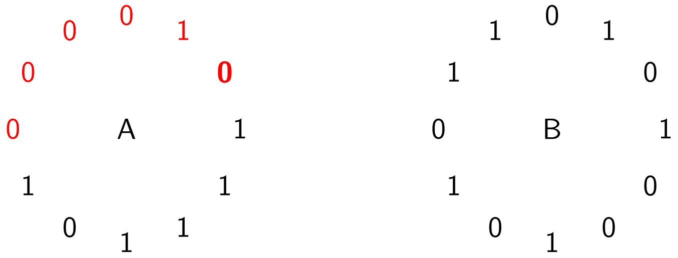




$A = 000010111101$ $B = 001010110101$

Cycle-joining example

Let $n = 6$.



$A = 000010111101$ $\quad B = 001010110101$

A cycle-join $B = \mathbf{0}$

Cycle-joining example

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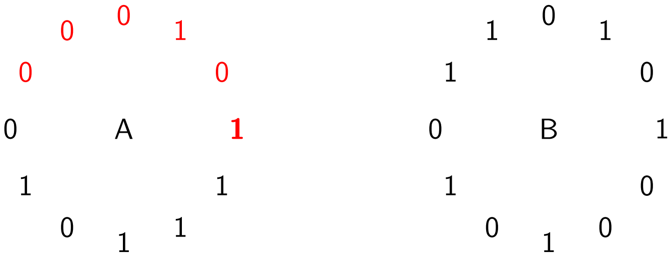

$A = 000010111101$ $B = 001010110101$

A cycle-join $B = 01$**1**

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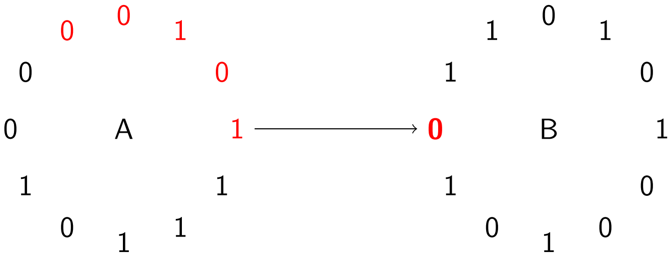

$A = 000010111101$ $\quad$ $B = 001010110101$

A cycle-join $B = 01\mathbf{0}$

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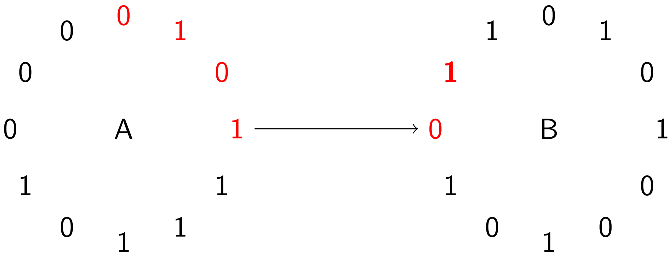

$A = 000010111101$ $B = 001010110101$

A cycle-join $B = 010\mathbf{1}$

Cycle-joining example

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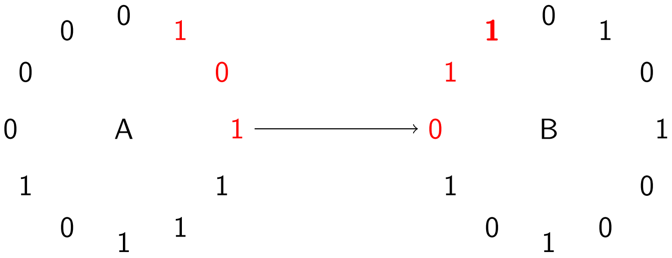
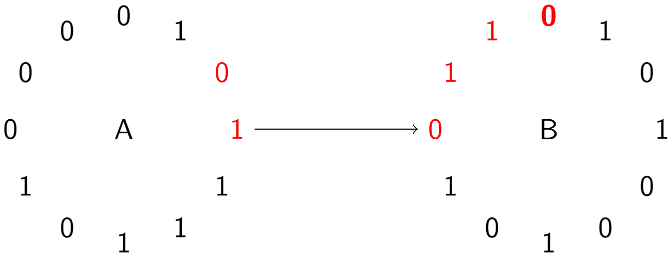


$$A = 000010111101 \qquad B = 001010110101$$

$$A \text{ cycle-join } B = 0101\mathbf{1}$$

Cycle-joining example

Let $n = 6$.



$$A = 000010111101 \qquad B = 001010110101$$

A cycle-join $B = 01011\mathbf{0}$

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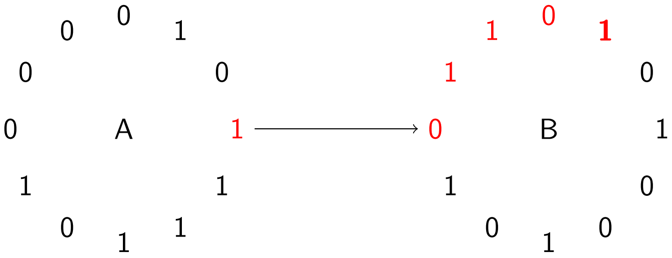
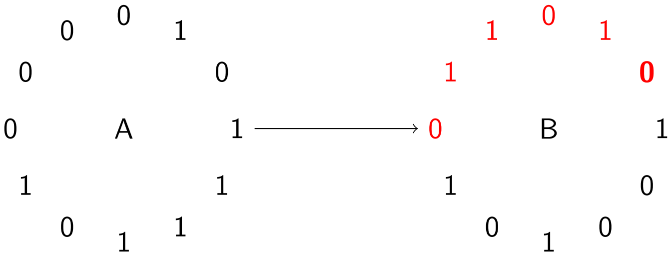


$A = 000010111101$ $B = 001010110101$

A cycle-join $B = 010110\mathbf{1}$

Cycle-joining example

Let $n = 6$.



$A = 000010111101$ $\quad$ $B = 001010110101$

A cycle-join $B = 01011010$

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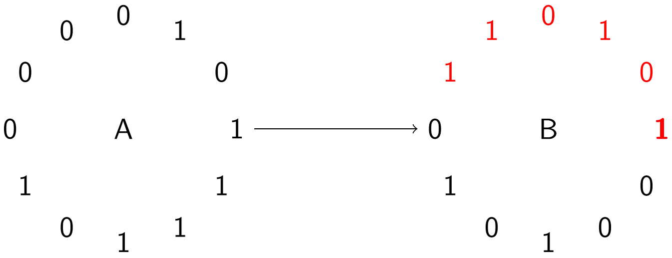

$A = 000010111101$ $B = 001010110101$

A cycle-join $B = 01011010\mathbf{1}$

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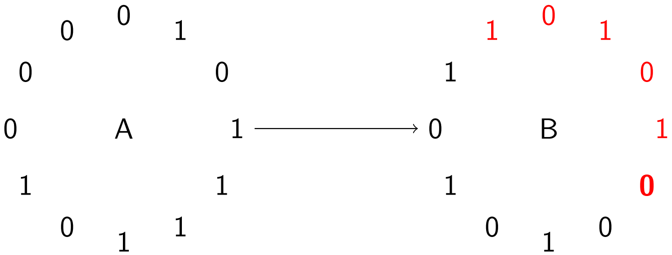

$A = 000010111101$ $\qquad$ $B = 001010110101$

A cycle-join $B = 010110101\mathbf{0}$

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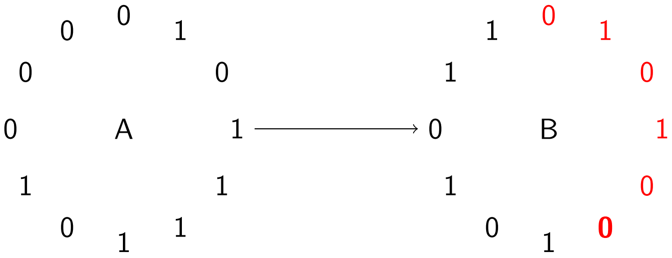

$A = 000010111101$ $\quad$ $B = 001010110101$

A cycle-join $B = 01011010100$**0**

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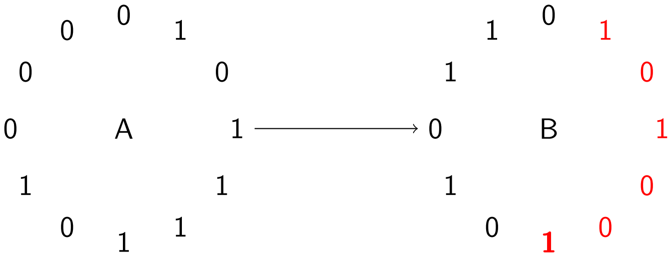

$A = 000010111101$ $B = 001010110101$

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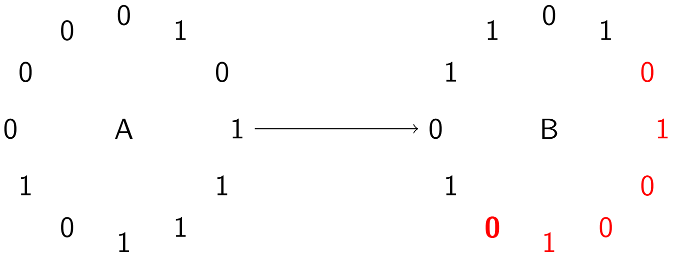

$$A = 000010111101 \qquad B = 001010110101$$

$$A \text{ cycle-join } B = 01011010100\mathbf{10}$$

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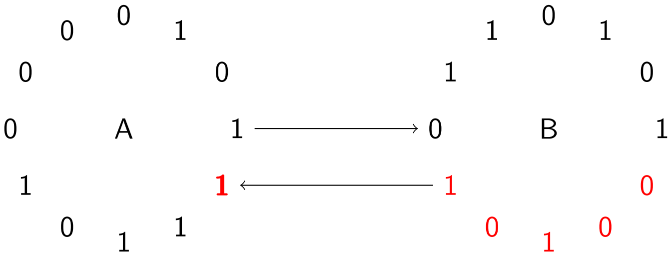

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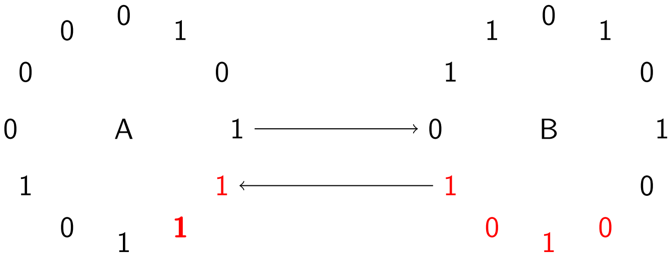

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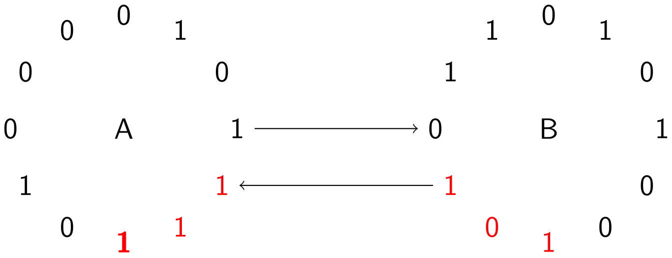

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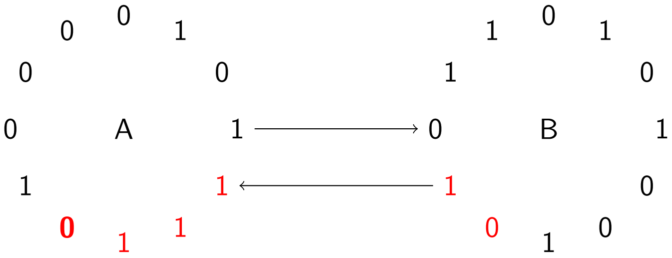

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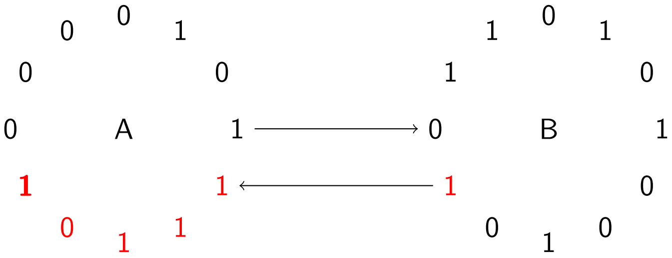

$$A = 000010111101 \qquad B = 001010110101$$

$$A \text{ cycle-join } B = 010110101001011110\mathbf{1}$$

Cycle-joining example

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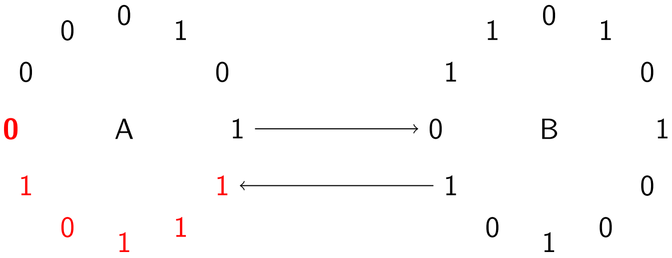

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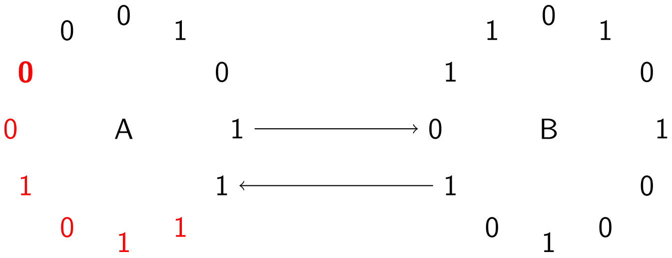

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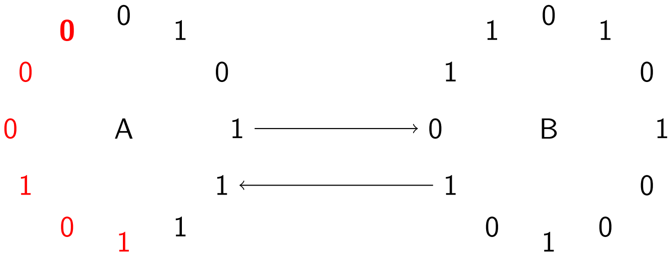

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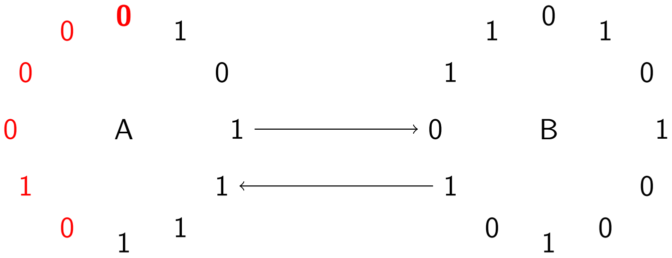

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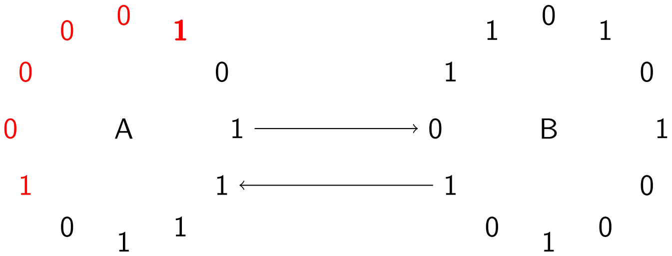

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- For example, the string 001011 is both a necklace and bracelet, and the string 001101 is a necklace but not a bracelet.

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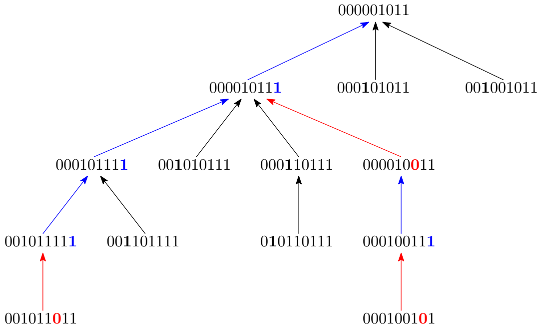
- They did not explicitly give an algorithm to construct such an $\mathcal{OS}(n)$.

Parent rule

Let $\alpha = \mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n$ be an asymmetric bracelet of length n.

- first1(α) be the necklace $\mathtt{a}_1\cdots\mathtt{a}_{i-1}\mathbf{0}\mathtt{a}_{i+1}\cdots\mathtt{a}_n$, where i is the index of the first 1 in α;
- last1(α) be the necklace in the necklace class of $\mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_{n-1}\mathbf{0}$;
- last0(α) be the necklace $\mathtt{a}_1\cdots\mathtt{a}_{j-1}\mathbf{1}\mathtt{a}_{j+1}\cdots\mathtt{a}_n$, where j is the index of the last 0 in α.

$$\text{par}(\alpha) = \begin{cases} \text{first1}(\alpha), & \text{if first1}(\alpha) \text{ is asymmetric;} \\ \text{last1}(\alpha), & \text{if first1}(\alpha) \text{ is not asymmetric and last1}(\alpha) \text{ is asymmetric;} \\ \text{last0}(\alpha), & \text{otherwise.} \end{cases}$$



Successor-rule g to construct an $\mathcal{OS}(n)$ of length L_n

- Let $\mathbf{A}(n)$ denote the set of length-n asymmetric bracelets.
- Let $\mathbf{S}(n)$ denote the set of all rotations of strings in $\mathbf{A}(n)$.

Let $\alpha = \mathtt{a}_1\mathtt{a}_2\cdots\mathtt{a}_n \in \mathbf{S}(n)$ and let

- $\beta_1 = 0^{n-i}\mathbf{1}\mathtt{a}_2\cdots\mathtt{a}_i$ where i is the largest index of α such that $\mathtt{a}_i = 1$ (first 1);
- $\beta_2 = \mathtt{a}_2\mathtt{a}_3\cdots\mathtt{a}_n\mathbf{1}$ (last 1);
- $\beta_3 = \mathtt{a}_j\mathtt{a}_{j+1}\cdots\mathtt{a}_n\mathbf{0}1^{j-2}$ where j is the smallest index of α such that $\mathtt{a}_j = 0$ and $j > 1$ (last 0).

Let

$$g(\alpha) = \begin{cases} \overline{\mathtt{a}}_1, & \text{if } \beta_1 \text{ and first1}(\beta_1) \text{ are in } \mathbf{A}(n); \\ \overline{\mathtt{a}}_1, & \text{if } \beta_2 \text{ and last1}(\beta_2) \text{ are in } \mathbf{A}(n), \text{ and first1}(\beta_2) \text{ is not in } \mathbf{A}(n); \\ \overline{\mathtt{a}}_1, & \text{if } \beta_3 \text{ and last0}(\beta_3) \text{ are in } \mathbf{A}(n), \text{ and neither first1}(\beta_3) \text{ nor last1}(\beta_3) \text{ are in } \mathbf{A}(n); \\ \mathtt{a}_1, & \text{otherwise.} \end{cases}$$

Theorem 4

The function g is a successor rule that generates an $\mathcal{OS}(n)$ with length L_n for the set $\mathbf{S}(n)$ in $O(n)$-time per bit using $O(n)$ space.

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- See `https://debruijnsequence.org/db/orientable` for a complete implementation of the successor rule.