Optimal-Time Queries and Constructions of RLBWT in BWT-runs Bounded Space

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Optimal-Time Queries and Constructions RLBWT in BWT-runs Bounded Space

- A summary of two papers presented at ICALP
  - 2021: Optimal-Time Queries on BWT-runs Compressed Indexes
  - 2022: An Optimal-Time RLBWT Construction in BWT-Runs Bounded Space

- A key element common to both papers is an efficient bipartite graph representation called LF-interval graph in RLBWT.

- Structure of the Talk:
  - First Half: Focus on Queries
  - Second Half: Focus on Constructions

Main Topic: Compressed Information Processing of Strings with Many Repetitions

- Recently, strings characterized by many repetitions have become widespread in both research and industrial applications.

  Ex: Genome sequences, version-controlled documents, and more

- Compressed information processing of these repetitive strings is a central research topic in string processing.

- Several formats have been developed for this type of data, including grammars and LZ-type compressions.

- We focus on RLBWT, a run-length compressed version of Burrows-Wheeler Transform (BWT) in this talk.

The Burrows-Wheeler Transform (BWT):

A permutation of a string T created by sorting all the circular shifts of T and taking the last column L of the resulting matrix.

- Key feature of BWT: Identical characters tend to cluster together, facilitating a compression
- LF mapping:
  LF[i] = (the number of characters smaller than L[i] in F) + (the rank of L[i] at position i in L)
- LF[i] returns the position in the sorted circular shifts of T obtained
  by moving the last character of the i-th circular shift to its beginning
- Property of LF: (A) it defines a one-to-one correspondence between column L and column F

(B) It maps positions with consecutive characters in L to the consecutive positions in F
(i.e., if L[i]=L[i+1], then LF[i+1]=LF[i]+1 holds)

T=babababaab$

Rotation

circular shifts of T

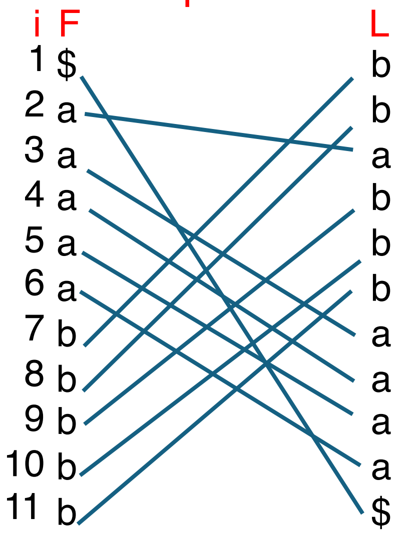
1	babababaab$
2	ababababaab$b
3	bababaab$ba
4	ababaab$bab
5	babaab$baba
6	abaab$babab
7	baab$bababa
8	aab$bababab
9	ab$babababa
10	b$babababaa
11	$babababaab

Sort

SA	F … L:BWT
11	$babababaab
8	aab$bababab
9	ab$babababa
6	abaab$babab
4	ababaab$bab
2	abababaab$b
10	b$babababaa
7	baab$bababa
5	babaab$baba
3	bababaab$ba
1	babababaab$

i	SA	sorted circular shifts
1	11	$babababaab
2	8	aab$bababab
3	9	ab$babababa
LF[8] → 4	6	abaab$babab
5	4	ababaab$bab
6	2	abababaab$b
7	10	b$babababaa
i → 8	7	baab$bababa
9	5	babaab$baba
10	3	bababaab$ba
11	1	babababaab$

One-to-one correspondence



Backward Search Using LF-mapping : Find the SA-interval [s,t] of pattern P on L

① Initialize [s,t] = [1,|L|] and h = |P|

② Find the first and last occurrence positions [k,ℓ] of character P[h] in the interval [s,t] on L

- Rank and select on L are used for computing [k,ℓ]

③ Compute s'=LF[k] and t'=LF[ℓ] using LF-mapping

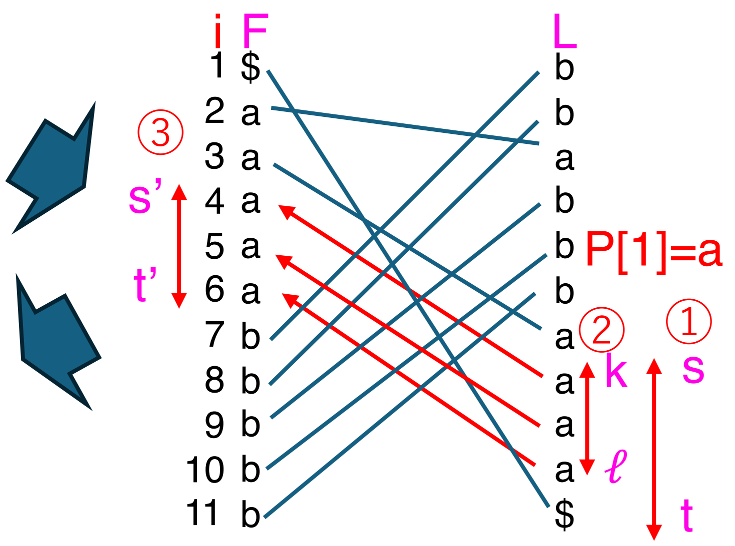
- Every element j ∈ [s',t'] satisfies the condition that suffix P[h..|P|] is a prefix of suffix T[SA[j]..|T|]

④ Update s=s', t=t', h=h–1, and go to step① if s ≦ t or h>0 hold

- Complexity: O(|P| log σ) time and O(|T| log σ) bits of space (σ : alphabet size)

Input
P=aba
h=1
[s,t]=[8,11]

Output
[s',t']=[4,6]



i	SA	sorted circular shifts
1	11	$babababaab
2	8	aab$bababab
3	9	ab$babababa
4	6	abaab$babab
5	4	ababaab$bab
6	2	abababaab$b
7	10	b$babababaa
8	7	baab$bababa
9	5	babaab$baba
10	3	bababaab$ba
11	1	babababaab$

Recovery of Occurrence Positions of Pattern P in T Using Suffix Array

- Backward search determines the SA-interval [s,t] on L that corresponds to pattern P.
- Once [s,t] is established, the positions of P in T can be computed using the suffix array SA.

  - $p = SA[j]$ for $j \in \{s, s+1, \ldots, t\}$

- Implementation detail: Suffix array is sampled and kept in memory for space efficiency
- If $|T|/\log|T|$ positions are sampled, $O(occ \log\sigma \log|T|)$ time and $O(|T|)$ bits of space are used.
  (σ: alphabet size, occ: number of occurrences of P in T)

[s,t]=[4,6] :
Interval of P=aba on L

i	SA	sorted circular shifts
1	11	$bababaab
2	8	aab$bababab
3	9	ab$babababa
4	6	abaab$babab
5	4	ababaab$bab
6	2	abababaab$b
7	10	b$bababaaa
8	7	baab$bababa
9	5	babaab$baba
10	3	bababaab$ba
11	1	babababaab$

T=babababaab$

p=2,4,6

Run-Length Encoded BWT (RLBWT)

- BWT L=bbabbbaaaa$ → RLBWT L'=b2a1b3a4$1
- A run is defined as the maximum repetition of the same character.
- Key Property: The BWT's ability to cluster the identical characters makes the run-length encoding particularly effective
- This property will significantly improve compression ratios.
- Actually, RLBWT is particularly effective for highly repetitive strings

Ex: 1,000 human genomes of chromosome 19 (60GB) can be compressed to a size of 250MB

- Technical Challenge : How can we realize backward search on RLBWT and occurrence position recoveries within the compressed size of RLBWT?

Previous Result: Backward Search and Occurrence Position Recoveries on RLBWT [T.Gagie, G.Navarro, N.Prezza, SODA'18, J.ACM'20]

- The researchers introduced the following three steps:

1. SA-interval computation: Compute SA-interval [s,t] on L that corresponds to pattern P
   - O(|P|loglog(|T|/r)) time
2. Suffix array computation: Compute suffix array SA[s] for the first position s in [s,t]
   - O(|P| loglog(|T|/r)) time
3. Occurrence position recoveries: Recover occurrence positions of P in T using Φ^{-1}-function
   - Φ^{-1}-function takes SA[i] and returns SA[i+1]
   - O(occ loglog(|T|/r)) time

- Space: O (r log |T|) bits (r: number of runs in T)
- Time: O((|P| + occ)loglog(|T|/r)) is not optimal (i.e., O(|P|+occ)).
  (occ: the number of occurrences of P in T)
- This arises due to the use of the predecessor data structure for computing LF-mappings and ϕ^{-1}-functions.

- We will improve each of these three steps by introducing a novel data structure, achieving O(|P|+occ) time and O(r log |T|) bits of space.

LF-interval Graph: A bipartite graph representing LF-mapping on BWT

- The structure consists of two sets of nodes and two types of edges

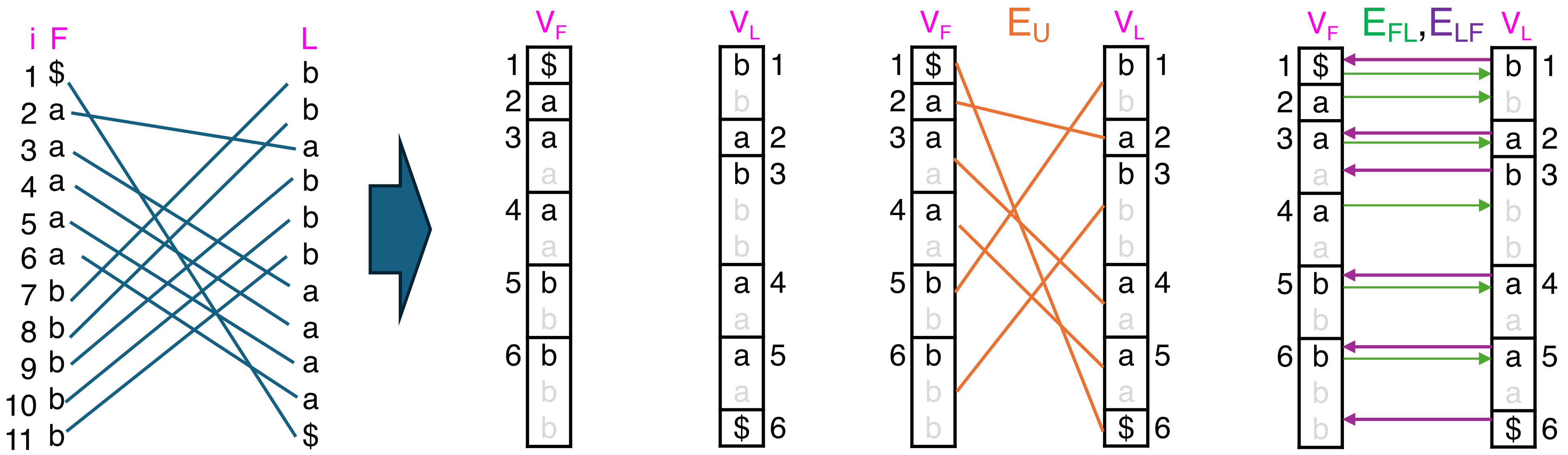
(I) Sets of nodes V_F, V_L: Each node in V_F and V_L represents a repetition in F and L, respectively.

(II) A set of undirected edges E_U: Represent LF-mapping between repetitions in nodes.

(III) Sets of directed edges E_{FL}, E_{LF}: Each edge in E_{FL} indicates the starting position of a repetition in V_F is included within the interval of the repetition of a node in V_L.

- E_{LF} is defined similarly.

- Two key properties of LF-interval graph:
  1. The number of nodes r' is bounded by O(r) (r: the number of runs in T)
  2. α-heavyness: The number of directed edges connecting to each node is bounded by O(α) (α: constant)



Backward Search on LF-interval Graphs : Find SA-interval [s,t] of pattern P on L

- LF-interval graph is traversed as performed during the backward search in the BWT
- k: the first position on L such that L[k]=c holds for (i) a given character c in P and (ii) a given SA-interval.
- u: the first node including position k on L in the repetition of u
- There are two important issues to be solved in backward search on LF-interval graph:

Q1: Which element d in the repetition of node u' on V_F corresponds to the s'-th element on F, where s' = LF[k]?

Q2: Which node on V_L contains the s'-th element in L in the repetition?

Figure for Q1

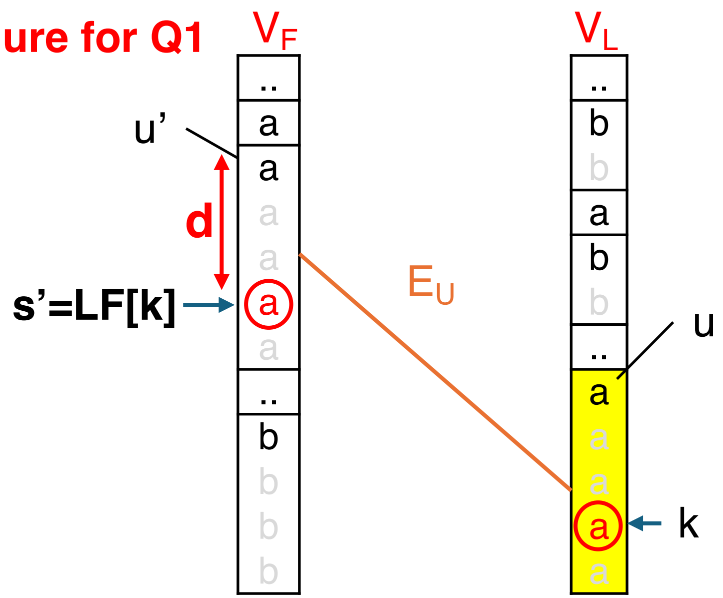
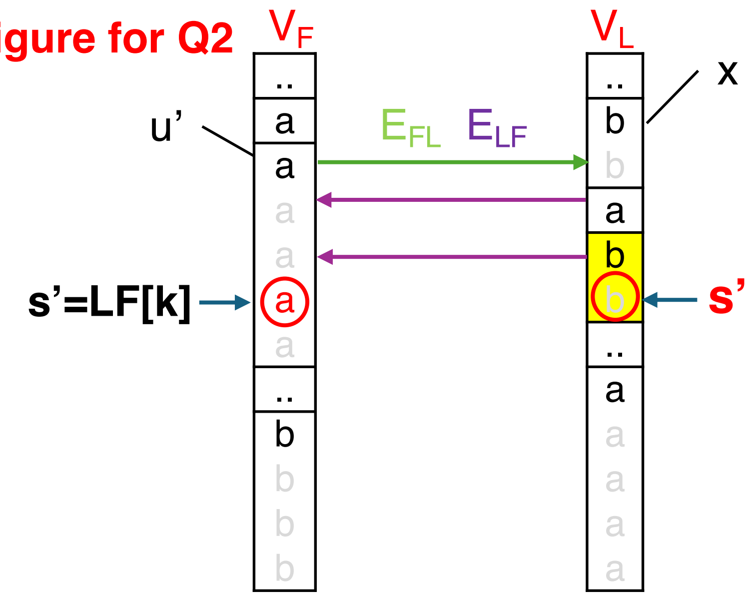


Figure for Q2



Backward Search on LF-interval Graphs : Find SA-interval [s,t] of pattern P on L

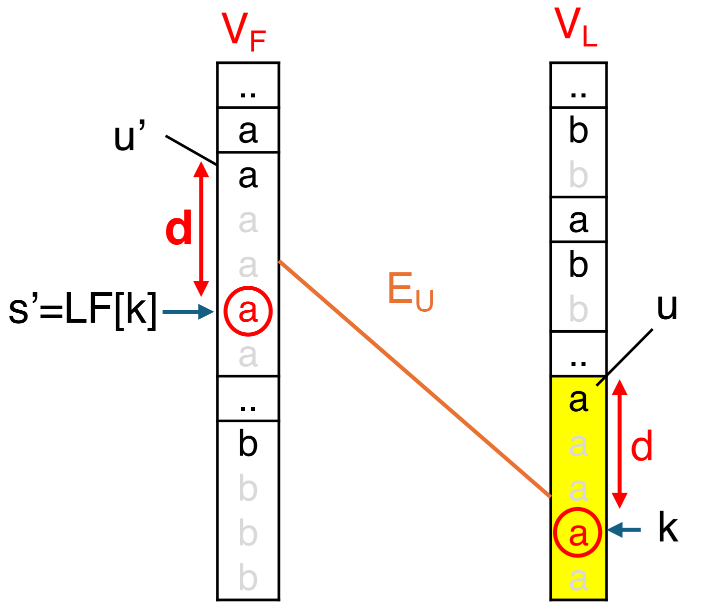
Q1: Which element d in the repetition of node u' on V_F corresponds to the s'-th element on F, where s' = LF[k]?

A1: Use the following property of LF-mapping:

Consecutive characters on u are mapped to consecutive ones on u'

- Thus, d is preserved in the two repetitions of nodes u and u' connected by an undirected edge
- The d-th element in the repetition of u' on V_F are computed from the same d-th element in the repetition of u on V_L

Figure for Q1



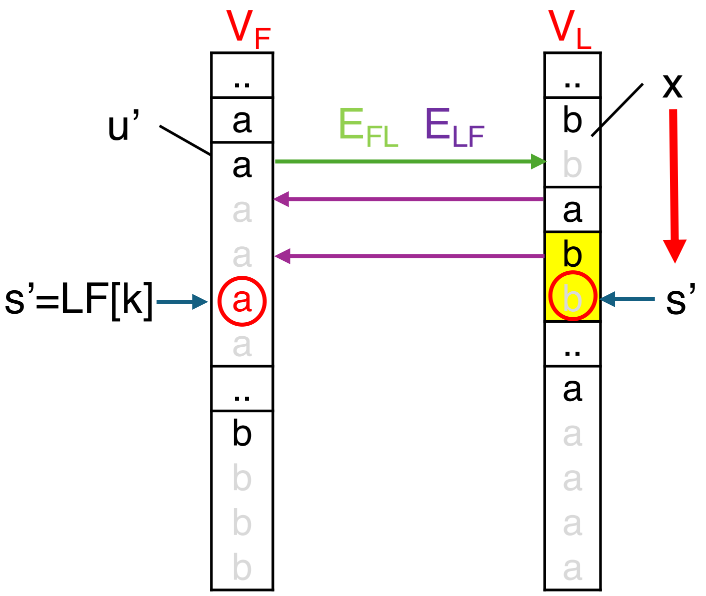
Backward Search on LF-interval Graphs : Find SA-interval [s,t] of pattern P on L

Q2: Which node on V_L contains the s'-th element in L in the repetition?

A2: Use this fact: Such node must connect to u' by a directed edge in E_{FL} or E_{LF}.

- Let x be the node connected to u' by a directed edge in E_{FL}
- A linear search starting from x on V_L can find a node including the s'-th element
- Use array A_L : which includes starting positions of the repetition of each node on V_L
- Computation time: O(α)
  - This efficiency is due to the number of nodes connected to u' by directed edges being O(α)

Figure for Q2



Compute suffix array SA[s] for the first position s in SA-interval [s,t]

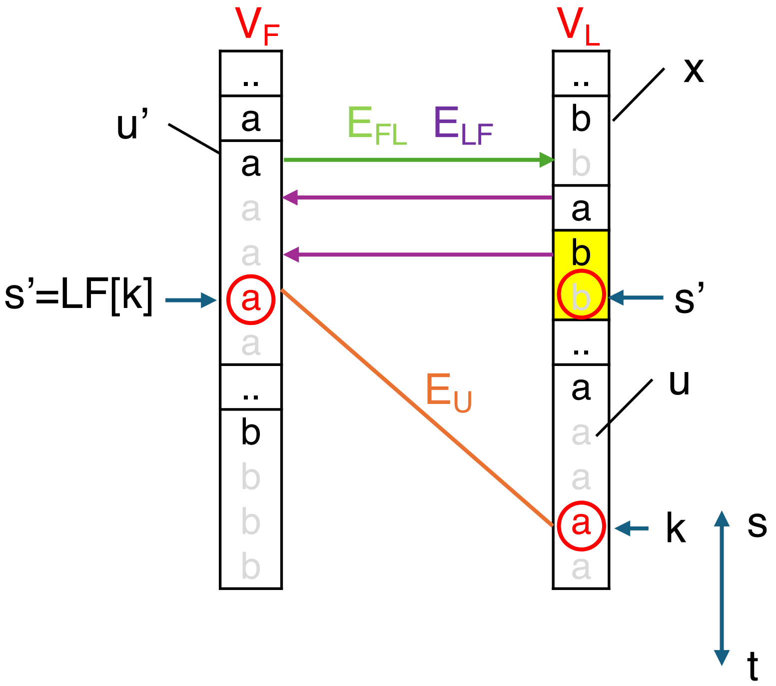
- Idea : Leverage the property of LF-mapping: SA[LF[i]]=SA[i]-1
- Thus, we can compute SA[s'] for the first position s' of the next SA-interval [s',t'] from SA[s] for the first position s of the current SA-interval [s,t].

- Compute SA[s'] as follows:

Case (i): If c=L[k] corresponds to the first character of the repetition of u, $SA[s'] = SA_{SAMP\_F}[u] - 1$

Case (ii): Otherwise, SA[s'] = SA[s] - 1

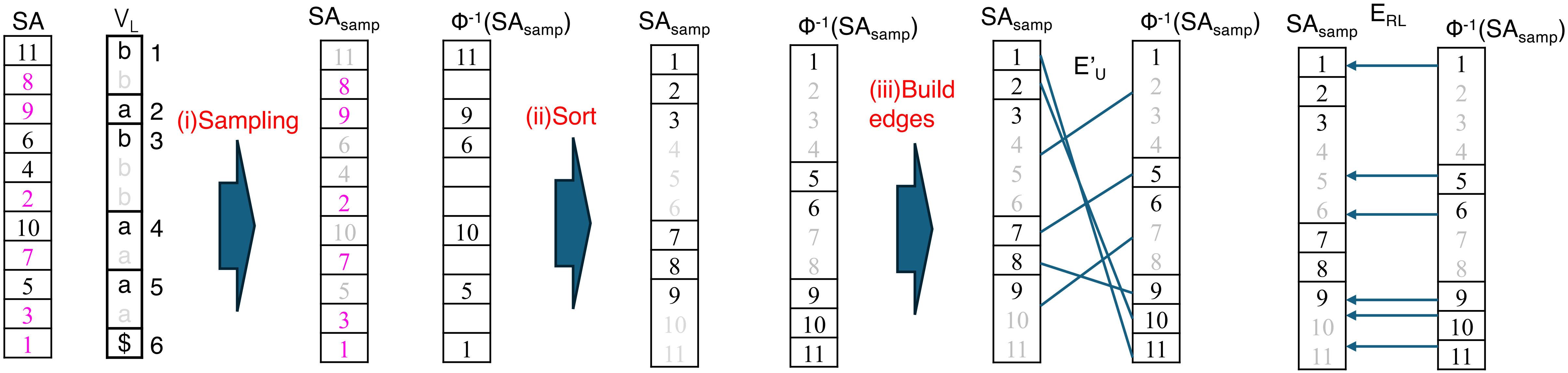
- Array SA_{SAMP_F} : sampled SA according to the starting position of the repetition of each node in V_L



- The computation is valid because case (i) must hold at the first iteration in the backward search.

Computing Φ^{-1}-function : Given SA[i], it returns SA[i+1]

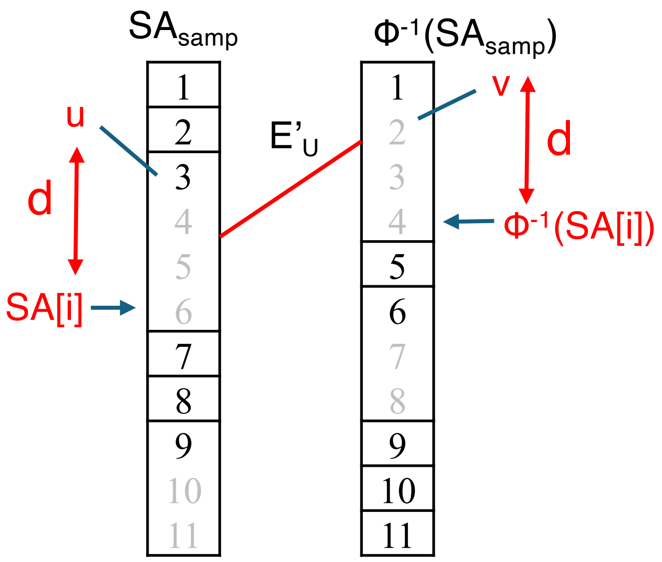
- Idea : (i) Build a partite graph that represents the relationship between input SA[i] and output SA[i+1] and (ii) compute Φ^{-1}-function on the graph
- Set of nodes SA_{samp} : Includes sampled SA according to the ending position of the repetition for each node on V_L
- Set of nodes $\Phi^{-1}(SA_{samp})$: Includes SA[i+1] if SA[i] is included in SA_{samp}
- Undirected edges E'_U : An edge connecting $SA_{samp}[i]$ to $\Phi^{-1}(SA_{samp})[j]$ indicates $\Phi^{-1}(SA_{samp})[j] = SA[i+1]$ holds
- Directed edges E_{RL} : Each position i in $\Phi^{-1}(SA_{samp})$ is included within the interval of a node in SA_{samp}.

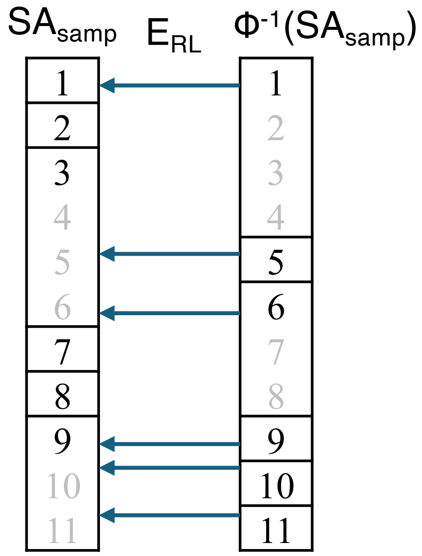


Computing Φ^{-1}-function : Given SA[i], it returns SA[i+1] (Cont.)

- For a given position i in SA-interval [s,t], let u be the node on SA_{SAMP} that contains SA[i] within the interval.
- Let v be the node connected to node u by an undirected edge in E'_U
- $\Phi^{-1}(SA[i])$ is computed by leveraging the following property:

  Each node in SA_{SAMP} represents the consecutive SA's values; The consecutive SA's values in each node are also mapped to $\Phi^{-1}(SA_{samp})$ as the same consecutive values.
- Can compute $\Phi^{-1}(SA[i])$ as follows: $\Phi^{-1}(SA[i])=\Phi^{-1}(SA_{samp})[v]+\underbrace{(SA[i]-SA_{samp}[u])}_{=d}$
- Detail: The next u corresponding to the computed $\Phi^{-1}(SA[i])$ is obtained using a linear search on SA_{samp}, starting from u' connected to v by the directed edge in E_{RL} (O(α) time)





Example for SA[i]=6:

u=3, v=1

$\Phi^{-1}(6)=\Phi^{-1}(SA_{samp})[1]+(6-SA_{samp}[3])$

$=1+(6-3)$

$=4$

Summary on Optimal-Time Backward Search and Occurrence Position Recoveries on RLBWT

1. SA-interval computation: Compute interval [s,t] on L that corresponds to pattern P
   - O(|P|loglog(n/r)) time → O(|P|) time
2. Suffix array computation: Compute suffix array SA[s] for the first position s in [s,t]
   - O(|P| loglog(n/r)) time → O(|P|) time
3. Occurrence position recoveries: Recover occurrence positions of P in T using Φ^{-1}-function
   - Φ^{-1}-function takes SA[i] and returns SA[i+1]
   - O(occ loglog(n/r)) time → O(occ) time

- O(|P|+occ) time and O(r log n) bits of space

- Optima-Time Construction of RLBWT

Extension of BWT (Review)

- BWT L' of string cT can be computed from the BWT L of string T throughout three steps.

① Relace the special character \$ on L by character c

② Insert the special character \$ into L at position k,
k is computed by the LF-formula:

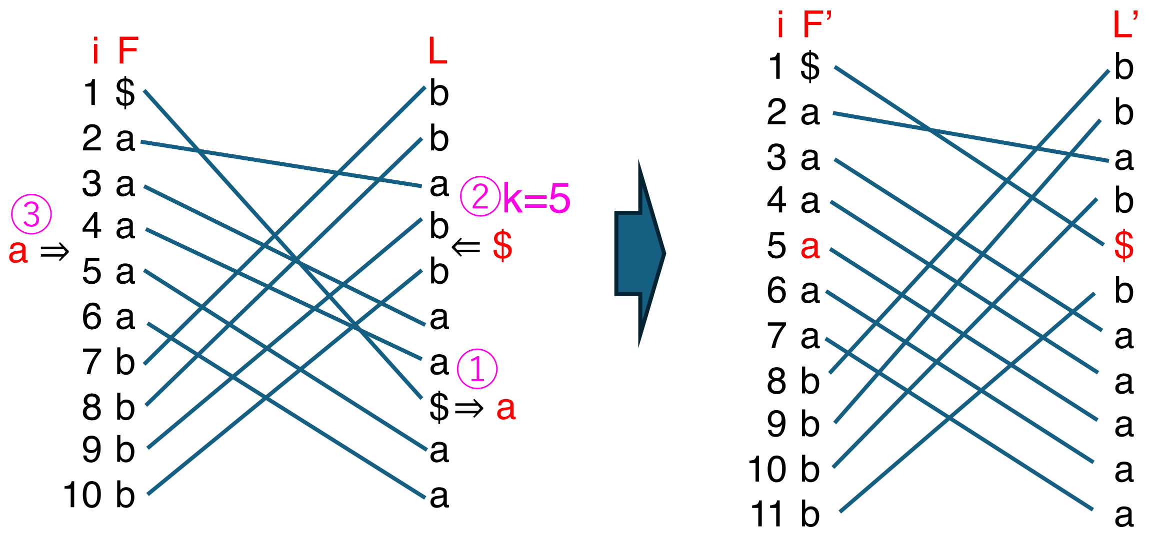
- $k = occ_{<}(L,c) + rank(L,j,c)$ for j such that L[j]=\$ holds

③ Insert character c into F at position k

- Update time: O(log |T|)
- We will update LF-interval graphs by leveraging this extension.

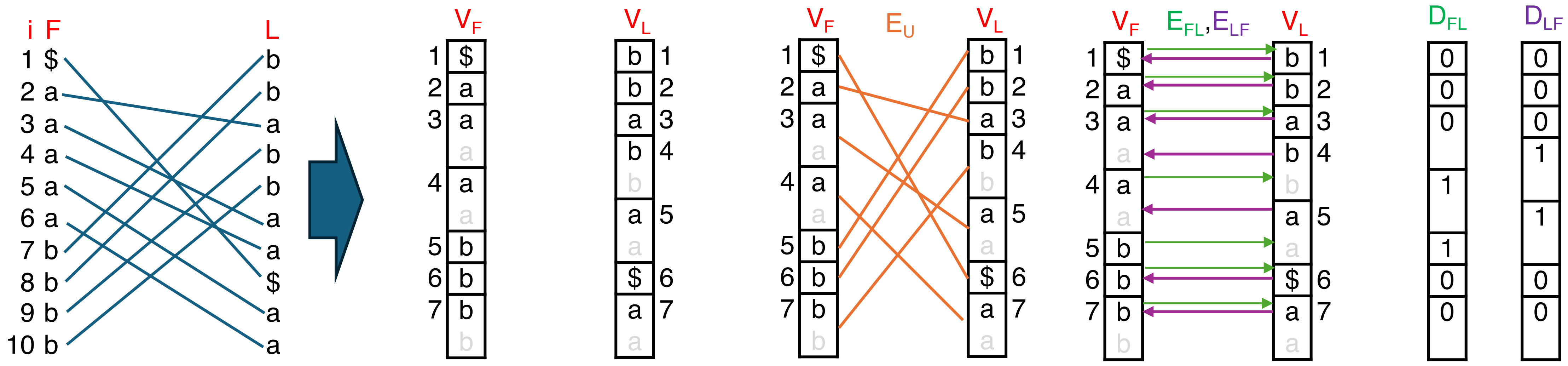
T=baaababab\$
c = a

Since L[8]=\$ and j=8,
k=occ<(L,a)+rank(L,8,a)
=1+4=5



LF-interval Graph and Additional Data Structures for Extensions

- The following data structures are added for extensions
- D_{FL}: Each element represents the difference between (i) the starting position of the repetition of node u on V_F and (ii) the starting position of the repetition of the node connected to u by the directed edge in E_{FL}
- D_{LF}: defined similarly to D_{FL}
- B-tree: used for identifying the insertion position of a new node on V_F
  - It keeps key-value pairs
  - key is a pair (c,v) for character c and node v on V_L
  - Value is the node u on V_F that is connected to v by an undirected edge in E_U
  - Given a key (c',v'), B-tree returns value u associated to the maximum key (c,v) satisfying c<c' or (c=c'∧v≤v')
- Order maintenance data structure for comparing nodes in V_L
- O(r' log n) bits of space in total (r': number of nodes)



Construction of RLBWT: Realizing the extension of BWT on LF-interval Graph

① **Replace-node:** (i) The node labeled \$ on V_L is replaced by a new node labeled c; (ii) A new node labeled c is inserted into an appropriate position on V_F (O(1) or O(log r) time)

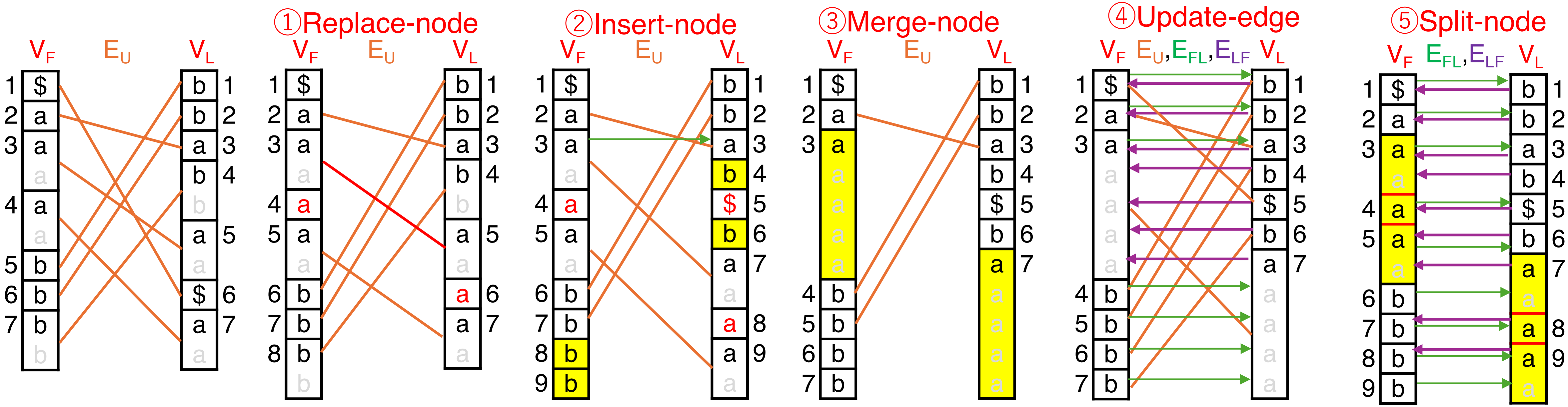
② **Insert-node:** A new node labeled \$ is inserted into an appropriate position on V_L (O(α) time)

③ **Merge-node:** If newly inserted nodes are adjacent to nodes with the same labels, they are merged (O(1) time)

④ **Update-edge:** Edges are updated appropriately. (O(α²) time)

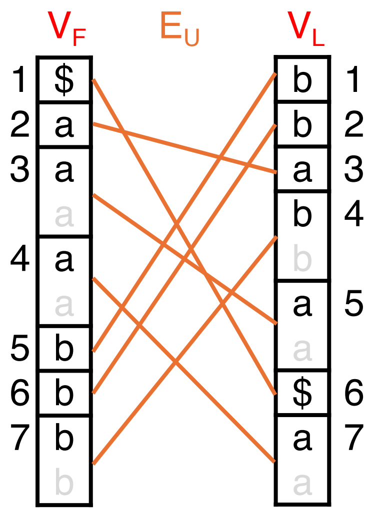
⑤ **Split-node:** Any node with at least α directed edges is split. (O(αr) time)

- Steps ①, ② and ⑤ are detailed in the following slides.

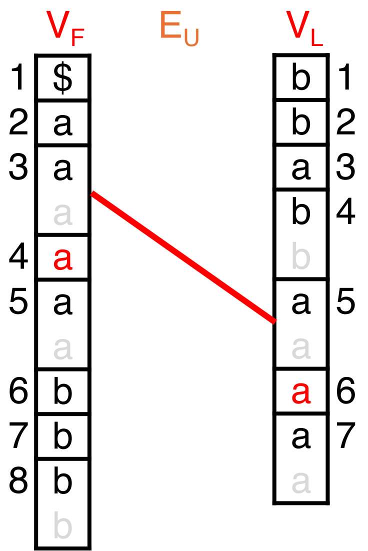


①Replace-node: (i) The node labeled \$ on V_L is replaced by a new node labeled c; (ii) A new node labeled c is inserted into an appropriate position on V_F

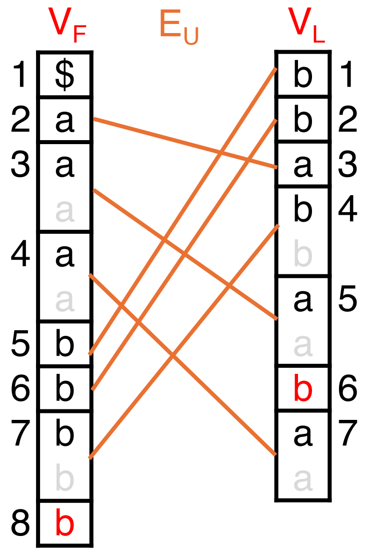
- How can we compute the position on V_F? There are two cases:
- Case ①: the new node v on V_L has the same label as either or both of adjacent nodes
- If the node is adjacent to the node v above and has the same label as v, the insertion position is below the node connected to v by an undirected edge in E_U.
- The other case is similarly computed. (O(1) time)
- Case ②: The insertion position is computed using B-tree. (O(log r) time)



Case ①: c=a



Case ②: c=b



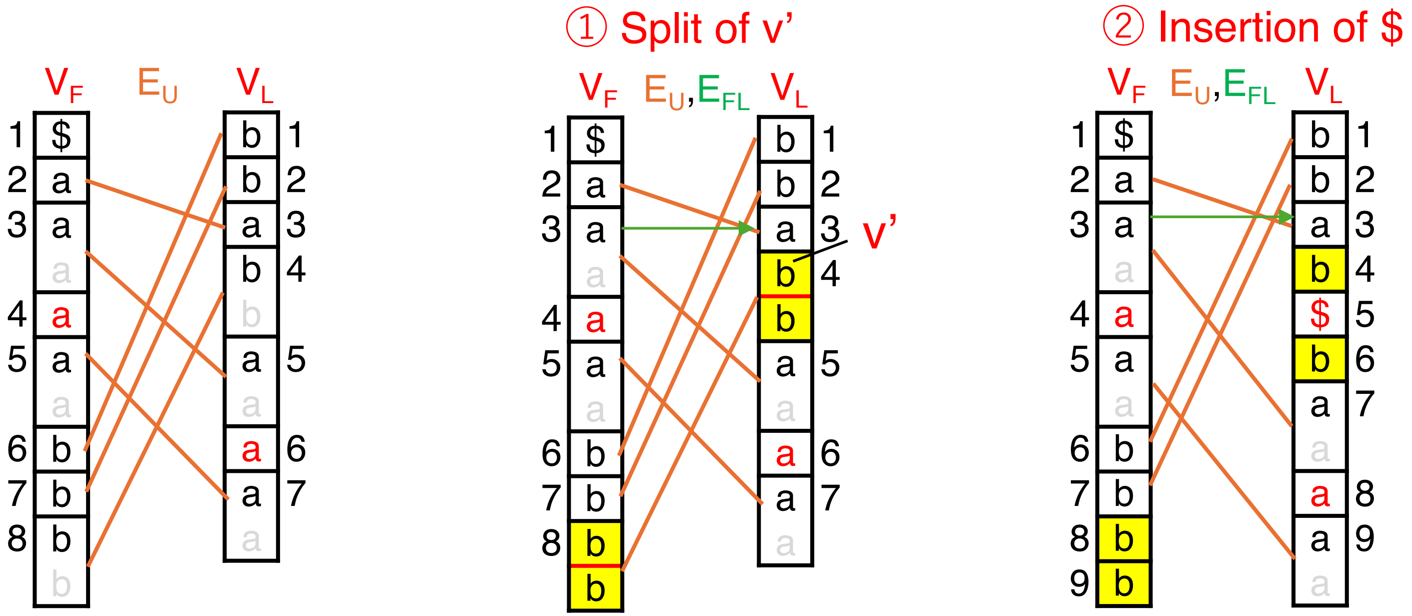
②Insert-node: Insert a new node labeled $ into an appropriate position on V_L

- Let v' be the node on V_L that includes the position of the inserted character c on V_F

① v' is then split

② a new node labeled $ is inserted between the split nodes.

- Computation time: O(α)

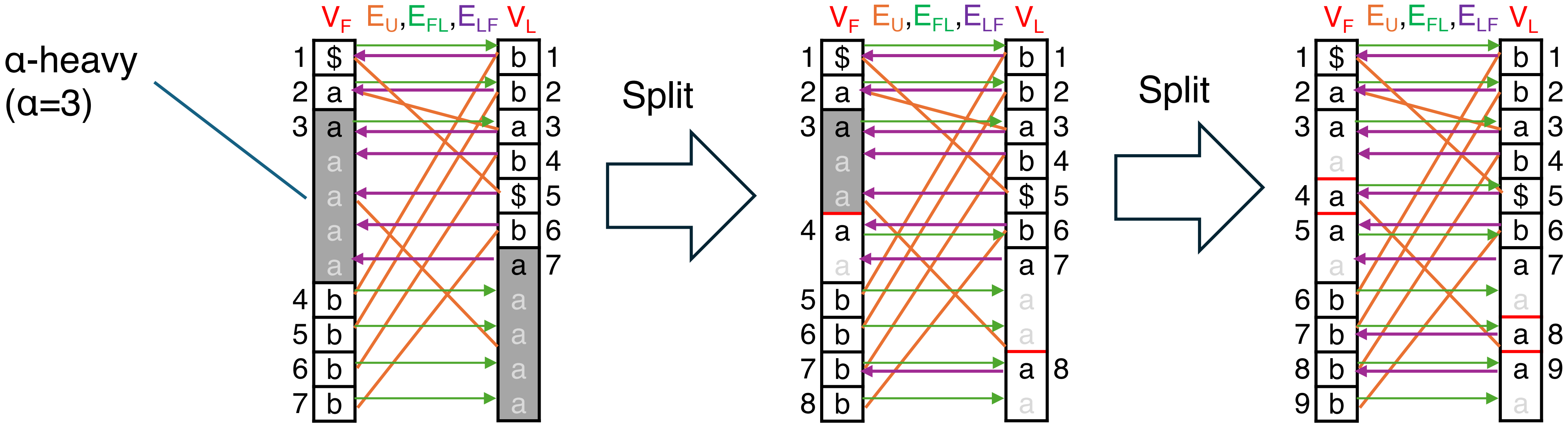


⑤Split-node: Any node with directed edges more than α (α-heavy) is split.

- Splitting nodes continues until the number of directed edges connected to each node is no more than α.
- Computation time per split: $O(\alpha)$
- The total number of split nodes: $O(r)$

Reason:

- The number of directed edges after m splits of nodes is at least $\lfloor \alpha/2 \rfloor m$.
- Meanwhile, the number of directed edges after m splits of nodes is $r + 2m$.
- Solving $\lfloor \alpha/2 \rfloor m \leq r+2m$ yields $m=O(r)$ for $\alpha \geq 16$.



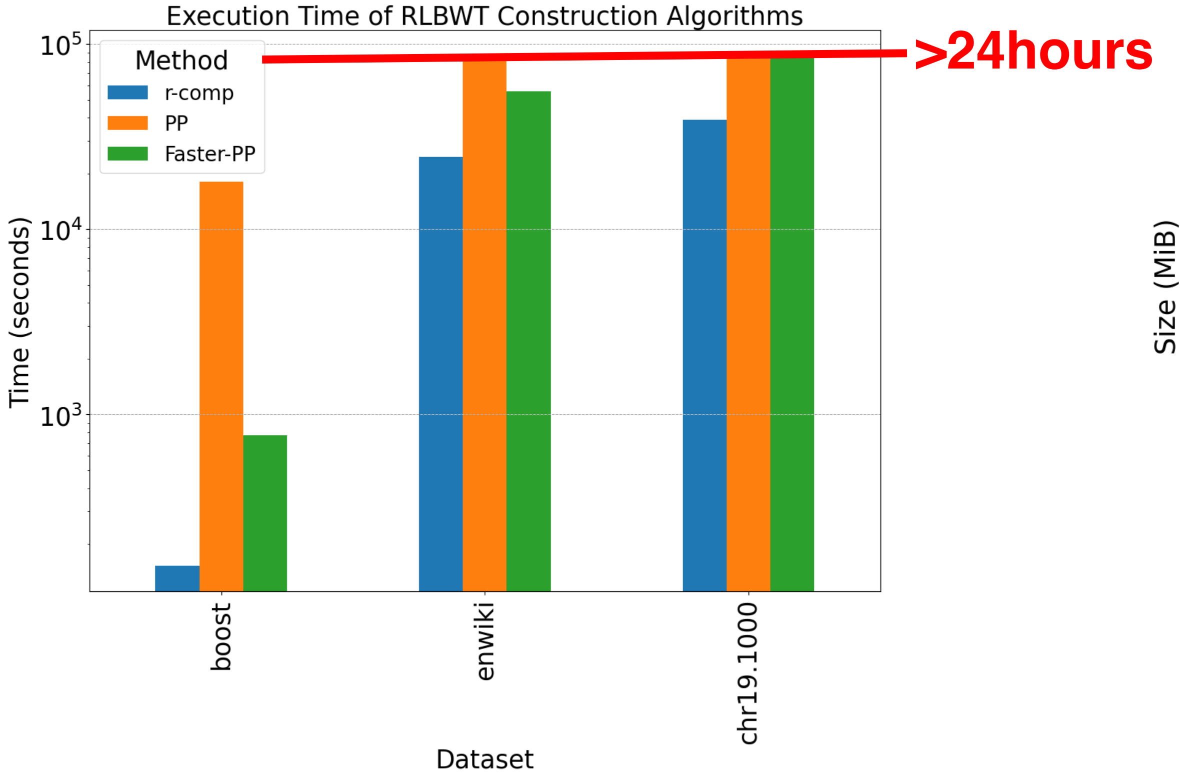
Experimental Results on A Large Dataset

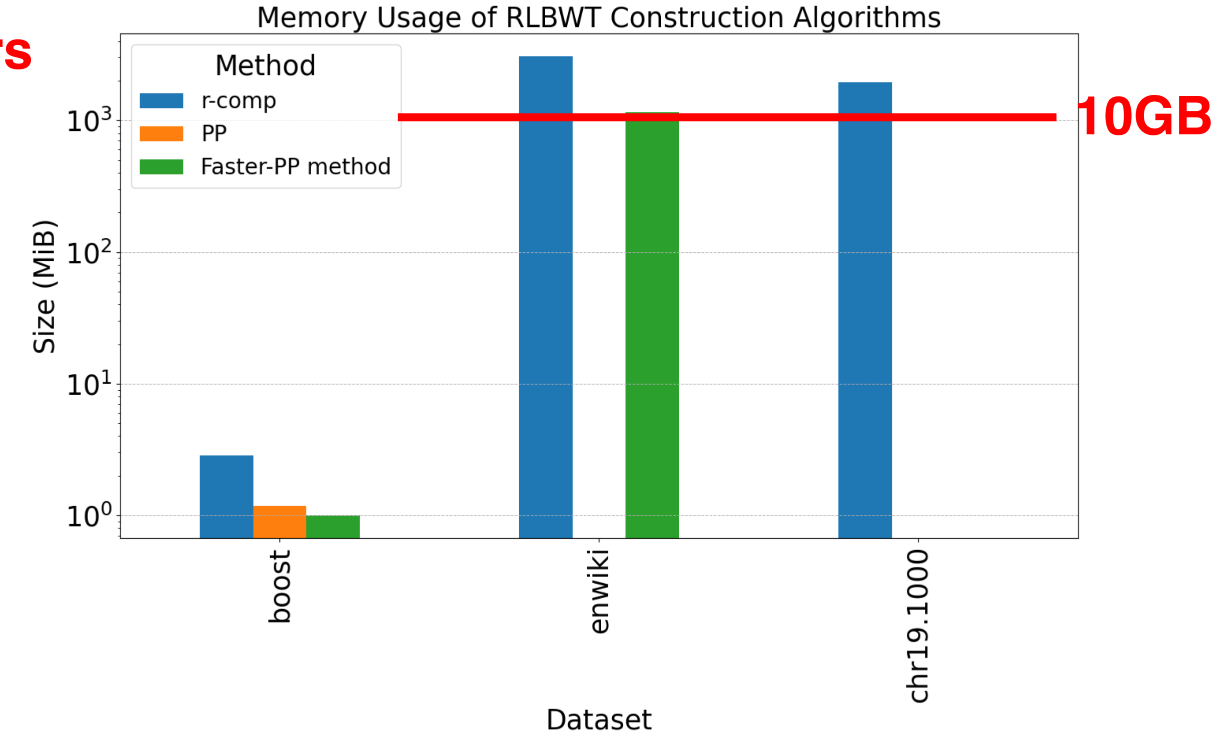
- R-comp (this study) is compared to

1. PP: A.Policriti and N.Prezza, 2018
2. Faster-PP: T. Ohno et al., 2018.

Dataset

String	σ	ITI [10^3]	r [10^3]
boost	96	1,073,769	65
enwiki	207	37,849,201	70,190
chr19.1000	5	59,125,169	45,143





Summary on Optimal-Time Construction of RLBWT

- Construction time for string T at each step is summarized as follows:

① Replace-node: $O(|T|+r \log r)$

② Insert-node: $O(|T|\alpha)$

③ Merge-node: $O(|T|\alpha)$

④ Update-edge: $O(|T|\alpha^2)$

⑤ Split-node: $O(r\alpha)$

- Total construction time: $O(|T|\alpha^2+r\alpha \log r)$
- $O(|T|)$ time holds for constant α and $r = |T|/\log|T|$ (satisfied for strings with many repetitions!)
- $O(r \log|T|)$ bits of space (because the total number of split nodes is $O(r)$)

Summary of This Talk

- We have presented optimal-time queries and constructions of RLBWT in BWT-runs Bounded Space
- A key element is an efficient bipartite graph representation called LF-interval graph in RLBWT.

- Backward search and occurrence position recoveries
  - Complexity: $O(|P|+occ)$ time and $O(r \log |T|)$ bits of space
- Construction
  - Complexity: $O(|T|)$ time and $O(r \log |T|)$ bits of space

- Take-home message from this talk:

Bipartite graphs are useful for efferently representing LF-mapping in BWT!